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THÈSE

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Étude d'un problème d'interaction fluide–structure : Modélisation, Analyse, Stabilisation et Simulations numériques.

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Table des matières

Introduction

Préambule

Contexte

Ce travail de thèse porte sur l'analyse, le contrôle et la simulation numérique des équations aux dérivées partielles. Plus précisément, nous nous sommes intéressés à la modélisation, l'analyse, la stabilisation et la simulation numérique d'un système d'interaction fluide–structure. L'étude d'un tel système, en plus d'appartenir à un domaine des mathématiques appliquées en plein essor, répond à des enjeux très variés. Par exemple, une application de ce type de problème est l'étude du réseau sanguin [102, 103, 131, 64], le fluide étant le sang et la structure le vaisseau sanguin. Dans ce cas, la structure contient le fluide.

Dans la suite, nous nous intéressons plutôt à des applications industrielles. On les trouve dans des domaines variés tels que l'automobile, la construction navale ou l'aéronautique. Dans ces trois domaines, les industriels construisent des véhicules (voitures, bateaux, avions) qui doivent se mouvoir au travers d'un fluide (air ou eau). Pour cela, les véhicules doivent déplacer le volume de fluide devant eux, ce qui consomme de l'énergie et donc du carburant. La façon dont les caractéristiques du véhicule (géométrie, vitesse, actionneurs) interviennent sur l'écoulement de fluide et donc l'effort consenti par le véhicule n'est pas évident à déterminer. L'étude des interactions entre fluide et structure peut donc potentiellement améliorer la performance de ces véhicules. Notons que dans ces cas d'application, c'est la structure qui est contenue dans le fluide.

Le lecteur pourra trouver d'autres exemples de systèmes d'interaction fluide–structure dans [136, 22, 39, 38, 73, 124, 123, 141].

Dans la suite, nous nous intéressons à un système d'interaction fluide–structure dans lequel la structure peut se déformer et dépend d'un nombre fini de paramètres scalaires. On modélise le comportement de cette structure par l'application d'un principe des travaux virtuels, ce qui nous amène à considérer une dynamique décrite par une équation différentielle ordinaire non linéaire. Le fait d'avoir une équation représentant la dynamique d'une structure déformable dépendant d'un nombre fini de paramètres scalaires est original par rapport à la littérature.

Les thématiques abordées : Nous introduisons ci–dessous trois aspects que nous abordons dans ce mémoire.

— Tout d'abord, une étude de modélisation du système que l'on souhaite étudier est nécessaire pour se doter d'un système d'équations représentant les phénomènes physiques auxquels on s'intéresse. Ce système doit présenter une unique solution, au moins localement autour d'un état initial. Cette thématique sera abordée dans le Chapitre 1. Une fois ces équations déterminées, on peut les étudier et notamment traiter les points suivants.

— Lorsque l'on considère un système physique, il est également possible de considérer des

actionneurs capables d'agir sur le système, par exemple une gouverne d'avion agit sur l'écoulement d'air environnant en se braquant dans une direction. Une question naturelle est alors de déterminer la façon dont doit agir l'actionneur pour que le système ait le comportement souhaité. Par exemple, on peut se demander de quelle façon il faut incliner un aileron pour que l'avion ait la bonne trajectoire. L'étude de la façon d'agir des actionneurs peut être menée dans un cadre mathématique, on appelle cela un problème de contrôle. On utilise alors la modélisation du système que l'on a précédemment établie et on y ajoute les contributions représentant l'actionneur que l'on appelle le contrôle.
Un cas particulier de problème de contrôle consiste à considérer un état d'équilibre instable et à chercher à maintenir le système autour de cet état d'équilibre en utilisant un contrôle donné. On parle alors de problème de stabilisation. Nous étudierons cette problématique dans le Chapitre 2.

— Pour obtenir des résultats plus précis sur le comportement d'un système, on peut utiliser des algorithmes adaptés permettant de simuler par ordinateur le phénomène physique qui nous intéresse. On parle de simulation numérique. Les résultats approchés fournis par l'ordinateur servent à prévoir le comportement du système auquel on s'intéresse sans avoir à faire des tests expérimentaux généralement plus coûteux. On peut alors dimensionner et valider un système sur ordinateur sans avoir à le créer physiquement. Cette technique permet de réduire les coûts de développement, en particulier pour les produits industriels de haute technologie, et d'augmenter leurs performances. Cette thématique sera abordée dans le Chapitre 3.
Notons que nous calculerons numériquement la loi de commande que nous aurons établie dans le Chapitre 2 pour stabiliser le système autour d'un état d'équilibre stationnaire.

Le problème étudié

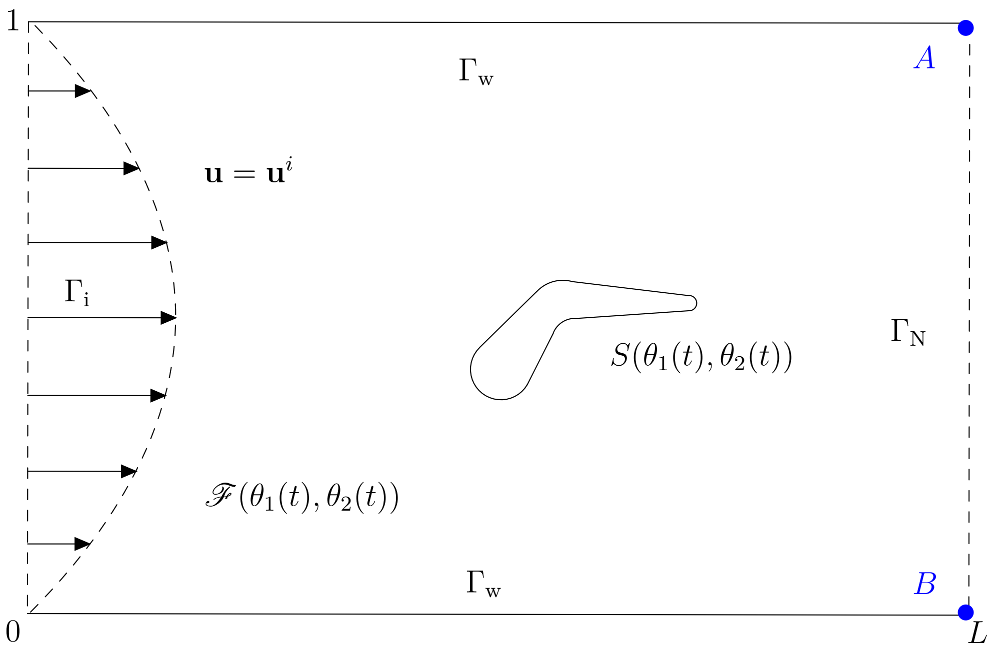


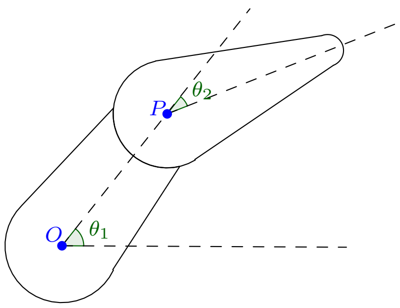
FIGURE 1 – La configuration du problème.

On s'intéresse dans ce mémoire au comportement d'une aile d'avion fixée à l'intérieur d'une veine de soufflerie. Un fluide (l'air) se déplace dans cette soufflerie et interagit avec la structure (l'aile d'avion). Pour simplifier notre étude, nous étudions une coupe de cette configuration, le problème est donc un problème 2D et la structure correspond à un profil donné de l'aile d'avion. La configuration étudiée est représentée sur la Figure 1.

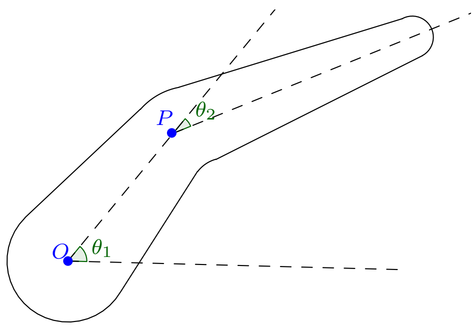
Nous notons $\Omega = (0, L) \times (0, 1)$ le domaine global de la veine de soufflerie. Le bord d'entrée de la soufflerie est noté $\Gamma_{\mathrm{i}} = \{0\} \times (0, 1)$, le bord de sortie $\Gamma_{\mathrm{N}} = \{L\} \times (0, 1)$ et les murs $\Gamma_{\mathrm{w}} = (0, L) \times \{0, 1\}$. L'écoulement d'air incident est noté $\mathbf{u}^i$, il correspond sur notre figure à un écoulement de Poiseuille.

Nous considérons deux mouvements admissibles pour l'aile d'avion, elle peut effectuer un mouvement de rotation autour d'un point O fixe dans le référentiel de la soufflerie. L'angle entre un axe de référence du profil et l'axe horizontal de la soufflerie est appelé l'assiette, nous le notons θ_1 par la suite. Il correspond à la rotation du profil.

Le deuxième mouvement considéré est l'actionnement d'un aileron à l'arrière du profil. Cet aileron est relié au corps du profil par l'intermédiaire d'une liaison pivot, il est donc en rotation autour d'un point P fixe dans le référentiel du profil. On note θ_2 l'angle de braquage de l'aileron. La géométrie de la structure que nous souhaitons étudier est représentée sur la Figure 2a. On considère par la suite qu'à tout instant le couple de paramètres (θ_1, θ_2) appartient à un domaine admissible de $\mathbb{R}^2$ noté $\mathbb{D}_\Theta$.



(a) Une structure avec une liaison pivot.



(b) Une structure se déformant continûment.

Figure 2 – La structure pivot et son approximation.

Le volume occupé par la structure dépend des deux paramètres θ_1 et θ_2, on le note donc $S(\theta_1, \theta_2) \subset \Omega$. Le fluide occupe le reste du domaine Ω, on note $\mathscr{F}(\theta_1, \theta_2) = \Omega \backslash S(\theta_1, \theta_2)$ le domaine du fluide. Remarquons que ce domaine dépend de l'état de la structure $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$ et peut donc varier au cours du temps. Il s'agit là d'un couplage géométrique entre le fluide et la structure. Il n'est donc pas évident de définir les espaces fonctionnels nécessaires à l'analyse. C'est une difficulté classique qui est inhérente aux problèmes d'interaction fluide–structure. Nous la résolvons en suivant une approche désormais classique [147, 34] : nous introduisons un difféomorphisme qui transporte le domaine mobile $\mathscr{F}(\theta_1, \theta_2)$ sur un domaine fixe au cours du temps.

Le modèle de structure représenté sur la Figure 2a est comparable à certains modèles simplifiés utilisés par les aérodynamiciens [57, 4, 95]. Il permet de rendre compte de phénomènes physiques critiques dans la conception et l'utilisation d'aéronefs. Nous donnons deux exemples de tels phénomènes concernant les ailes d'avion : la divergence statique [57, p.33] et le flottement, voir [57, p.81] et [4, 95].

La divergence statique est une rupture statique de la structure. Elle se produit lorsque l'écoulement incident dépasse une vitesse critique appelée vitesse de divergence entraînant des efforts aérodynamiques qui deviennent plus intenses que les forces de rappel de la structure.

Le flottement est un phénomène de résonnance entre le fluide et la structure. Des vibrations apparaissent au cours du temps. Pour des petites vitesses du fluide incident, ces vibrations sont amorties. Cependant, il exite une vitesse du fluide, appelée vitesse critique de flottement, à partir de laquelle ces vibrations s'amplifient. Ce phénomène, lorsqu'il se produit, a pour conséquence la destruction de la structure.

Pour étudier la divergence statique, on peut ne considérer que le degré de liberté θ_1. Pour étudier le flottement, il faut, en plus des degrés de liberté θ_1 et θ_2, ajouter un degré de liberté traduisant la translation verticale du point O auquel est rattaché la structure (pilonnement). Cela montre que le modèle que nous utilisons pour la structure a des applications en aérodynamique.

Notons que la Figure 2b représente une approximation de la Figure 2a avec un champ de vitesse continu dans la structure, nous donnerons plus de détails la concernant dans la section de modélisation.

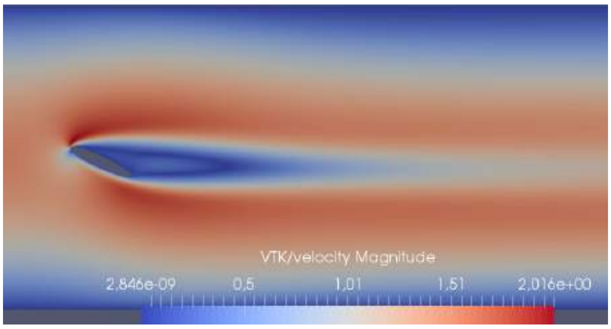
Motivations

Le dimensionnement des ailes d'un avion résulte d'un compromis entre les contraintes induites lors des différentes phases de vol de l'avion (décollage, croisière, atterrissage). Ce compromis fait que la géométrie de l'aile est sous optimale pour chacune de ces phases de vol. Ainsi, en faisant évoluer la forme des ailes au cours du vol, on peut l'adapter et donc améliorer les performances dans chacune des phases. Ceci permet notamment de réduire la consommation de carburant.

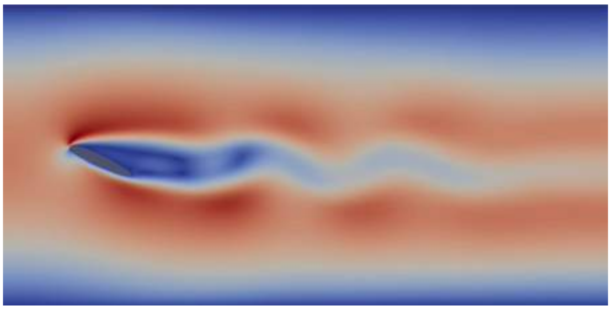
On appelle *morphing* les technologies qui améliorent les performances d'un véhicule en manipulant certaines de ses caractéristiques pour mieux l'adapter à son environnement [76, 12, 152, 91, 151]. Par adaptation des caractéristiques, on pense généralement à des changements importants de la géométrie, ce qui n'est pas le cas de la structure représentée sur la Figure 2a. Cependant, si l'on considère une structure constituée d'un nombre fini (≥ 2) de solides rigides, on peut obtenir des déformations plus complexes qui entrent dans le cadre du morphing. Si l'on souhaite étudier de telles structures, on peut alors utiliser les outils analytiques et numériques que nous mettons en place pour deux solides rigides. L'étude du modèle que nous proposons peut donc être vue comme un premier pas vers l'étude du morphing.

Dans ce mémoire, nous nous concentrons sur le contrôle actif de l'écoulement fluide grâce à la déformation de la structure. Pour la gamme de Reynolds que nous considérons ($\mathcal{R}e \sim 120$), la solution stationnaire du problème, représentée sur la Figure 3a, est instable. Ainsi, une perturbation de l'écoulement stationnaire fait apparaître des instabilités dans le sillage de la structure (voir Figure 3b) appelés allées de Von Karman. Ces instabilités dégradent les performances aérodynamiques de la structure. Nous cherchons donc un contrôle qui permet de les faire disparaître pour retrouver l'état stationnaire.

Un tel contrôle semble très difficile à mettre en œuvre dans le cadre de conditions de vol réelles, principalement à cause du nombre de Reynolds qui est beaucoup plus élevé (de l'ordre de plusieurs millions). Cependant, l'étude du contrôle actif à bas Reynolds est nécessaire avant de pouvoir l'adapter à des écoulements à haut nombre de Reynolds. On peut voir cette étude comme un premier pas vers le contrôle actif de l'écoulement d'air en vol, bien que la gamme de



(a) La solution stationnaire considérée.



(b) La solution perturbée.

FIGURE 3 – Les solutions stationnaire et perturbée ($\mathcal{R}e = 120$).

Reynolds étudiée ici ne corresponde pas aux conditions réelles d'un vol d'avion.

Modélisation

Revenons au modèle physique que nous avons présenté (Figure 2a). Comme introduit précédemment, nous notons $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$ les deux paramètres représentant la position de la structure. On note également $(\theta_{1,0}, \theta_{2,0}) \in \mathbb{D}_\Theta$ leurs valeurs initiales respectives.

Nous modélisons le fluide par les équations de Navier–Stokes incompressibles :

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x}) - \operatorname{div}\sigma_F(\mathbf{u},p)(t,\mathbf{x}) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in[0,T],\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\ \operatorname{div}\, \mathbf{u}(t,\mathbf{x}) = 0, & t\in[0,T],\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\ \mathbf{u}(t,\mathbf{x}) = 0, & t\in[0,T],\ \mathbf{x}\in\Gamma_{\mathrm{w}} \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in[0,T],\ \mathbf{x}\in\Gamma_{\mathrm{i}}, \\ \sigma_F(\mathbf{u},p)(t,\mathbf{x})\mathbf{n}(\mathbf{x}) = 0 & t\in[0,T],\ \mathbf{x}\in\Gamma_{\mathrm{N}}, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}), \end{cases} \tag{1}$$

où $\mathbf{u}$ et p sont respectivement la vitesse et la pression du fluide, $\sigma_F(\mathbf{u},p) = \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) - p\mathrm{I}$ est le tenseur des contraintes et $\nu > 0$ la viscosité du fluide. La normale unitaire sortante à Ω est notée $\mathbf{n}$.

Par la suite, des conditions d'adhérence sont imposées entre le fluide et la structure. Pour des raisons techniques liées à ces conditions d'adhérence, nous souhaitons que le champ de vitesse à l'intérieur de la structure soit suffisamment régulier en espace (voir hypothèse (6) ci–dessous). Or ce champ n'est même pas continu dans le cas de la structure de la Figure 2a que nous avons proposée. En effet, en regardant la Figure 2a, si l'on fixe $\theta_1 = 0$ et que l'on fait varier θ_2, alors la vitesse de la structure est discontinue à l'interface entre le premier solide au repos et le deuxième solide en rotation.

Pour avoir un champ de vitesse continu à l'intérieur de la structure, on décide d'approcher la structure de la Figure 2a par une structure se déformant de manière continue en espace, il faut donc "lisser" la déformation de la matière proche de l'interface entre les deux solides. Nous voudrions donc utiliser une structure comme celle représentée sur la Figure 2b.

La principale difficulté à l'utilisation d'une telle structure est que nous ne pouvons obtenir ses équations ni à partir des équations de l'élasticité linéaire, puisqu'elle ne dépend que de deux paramètres, ni des lois de Newton, puisqu'elle n'est pas un assemblage de solides rigides. Il faut donc faire un travail supplémentaire de modélisation pour pouvoir donner des équations à la structure de la Figure 2b.

On note $\mathbb{D}_\Theta$ un connexe ouvert borné de $\mathbb{R}^2$ qui représente le domaine admissible pour les paramètres θ_1 et θ_2. Nous considérons une fonction $\mathbf{X} : \mathbb{D}_\Theta \times S(0,0) \to S(\theta_1,\theta_2)$ qui à chaque couple de paramètres $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$ et à chaque point $\mathbf{y}$ dans la configuration de référence $S(0,0)$ associe le point $\mathbf{X}(\theta_1,\theta_2,\mathbf{y})$ correspondant à la même particule de matière dans la configuration $S(\theta_1,\theta_2)$. On suppose également que pour tout $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$, la fonction $\mathbf{X}(\theta_1,\theta_2,.)$ est un difféomorphisme de $S(0,0)$ vers $S(\theta_1,\theta_2)$. On note $\mathbf{Y}(\theta_1,\theta_2,.)$, allant de $S(\theta_1,\theta_2)$ vers $S(0,0)$, son difféomorphisme inverse. Ces fonctions sont représentées sur la Figure 4.

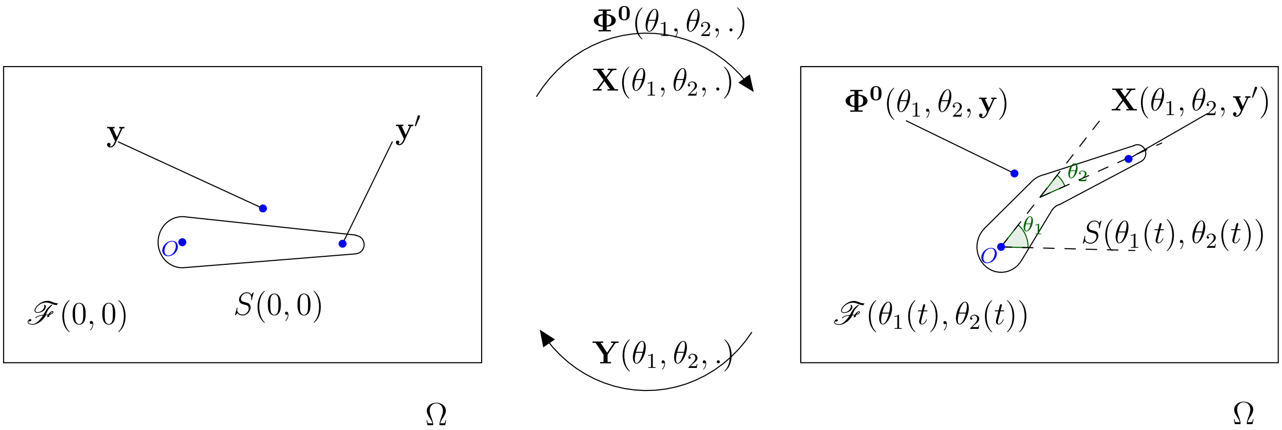


Figure 4 – Correspondance entre les configurations de référence et réelle.

Nous supposons les hypothèses suivantes sur $\mathbf{X}$:

Hypothèses de modélisation.

- $\mathbf{X} : \mathbb{D}_\Theta \times S(0,0) \to \Omega$. (2)
- $S(0,0)$ est un sous–ensemble borné fermé et simplement connexe de Ω. (3)
- Pour tout $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$, on a $S(\theta_1,\theta_2) = \mathbf{X}(\theta_1,\theta_2,S(0,0)) \subset \Omega$. (4)
- Pour tout $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1,\theta_2,.)$ est un $\mathscr{C}^\infty$ difféomorphisme de $S(0,0)$ sur son image $S(\theta_1,\theta_2)$. (5)
- La fonction $\mathbf{X}$ est $\mathscr{C}^\infty$ sur $\mathbb{D}_\Theta \times S(0,0)$. (6)
- Les fonctions $\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.)$ et $\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.)$ forment une famille libre dans $\mathbf{L}^2(\partial S(0,0))$ pour tout (θ_1,θ_2) dans $\mathbb{D}_\Theta$. (7)

L'hypothèse (2) donne le domaine de définition de $\mathbf{X}$, l'hypothèse (3) permet de restreindre l'étude à une catégorie de géométries, (4) porte sur $\mathbb{D}_\Theta$ qui est supposé suffisamment petit pour éviter tout contact entre la structure et $\partial\Omega$. L'hypothèse (5) implique que $\mathbf{X}(\theta_1,\theta_2,.)$ est un difféomorphisme, ce qui garantit l'existence de $\mathbf{Y}$. L'hypothèse (6) permet de définir les fonctions $\partial_{\theta_j}\mathbf{X}(\theta_1,\theta_2,.)$ et $\partial_{\theta_j\theta_k}\mathbf{X}(\theta_1,\theta_2,.)$. Finalement, (7) est utile pour définir les équations de la structure. Nous supposerons ces hypothèses vérifiées pour l'ensemble des travaux que nous présenterons dans ce manuscrit.

La vitesse d'une particule de matière dans la structure est donnée en coordonnées lagrangiennes par

$$\forall t \in [0,T], \quad \forall \mathbf{y} \in S(0,0), \quad \mathbf{v}_s(t,\mathbf{y}) = \dot{\theta}_1(t)\partial_{\theta_1}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}) + \dot{\theta}_2(t)\partial_{\theta_2}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}).$$

Le fluide satisfait une condition d'adhérence avec la structure, celle-ci peut être écrite en coordonnées eulériennes

$$\forall t \in [0,T], \quad \forall \mathbf{x} \in \partial S(\theta_1,\theta_2), \quad \mathbf{u}(t,\mathbf{x}) = \dot\theta_1 \partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\mathbf{x})) + \dot\theta_2 \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\mathbf{x})). \tag{8}$$

De plus, la dynamique de la structure est donnée par un principe des travaux virtuels (voir Chapitre 1 et [22, p. 14–17]) et peut être écrite sous forme matricielle comme suit

$$\forall t \in (0,T), \quad \mathcal{M}_{\theta_1,\theta_2} \begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = \mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) + \mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) + \mathbf{f_s}, \tag{9}$$

où les termes présents dans cette équation sont donnés par

$$\mathcal{M}_{\theta_1,\theta_2} = \begin{pmatrix} (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.))_S \\ (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.))_S \end{pmatrix} \in \mathbb{R}^{2\times 2}, \tag{10}$$

$$\mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) = \\ \begin{pmatrix} -(\dot\theta_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2,.)+2\dot\theta_1\dot\theta_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2,.)+\dot\theta_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2,.) \,,\, \partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.))_S \\ -(\dot\theta_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2,.)+2\dot\theta_1\dot\theta_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2,.)+\dot\theta_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2,.) \,,\, \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.))_S \end{pmatrix} \in \mathbb{R}^2, \tag{11}$$

où $(.,.)_S$ est le produit scalaire

$$(\boldsymbol{\Phi},\boldsymbol{\Psi})_S = \int_{S(0,0)} \rho(\mathbf{y})\boldsymbol{\Phi}(\mathbf{y})\cdot\boldsymbol{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{12}$$

avec $\rho(\mathbf{y}) > 0$ la masse volumique de la structure dans la configuration $S(0,0)$ et

$$\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) = \\ \begin{pmatrix} \displaystyle\int_{\partial S(\theta_1,\theta_2)} -\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}(\gamma_x)\cdot\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x \\ \displaystyle\int_{\partial S(\theta_1,\theta_2)} -\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}(\gamma_x)\cdot\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x \end{pmatrix} \in \mathbb{R}^2, \tag{13}$$

où $\mathbf{n}_{\theta_1,\theta_2}$ est la normale sortante à $\mathscr{F}(\theta_1,\theta_2)$ sur $\partial S(\theta_1,\theta_2)$.

L'équation (9) est complétée par les conditions initiales

$$\theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, \quad \dot\theta_1(0) = \omega_{1,0}, \quad \dot\theta_2(0) = \omega_{2,0}. \tag{14}$$

On suppose de plus que les données initiales appartiennent au domaine admissible pour la structure, c'est–à–dire $(\theta_{1,0},\theta_{2,0}) \in \mathbb{D}_\Theta$.

Le système d'équations que nous allons étudier est formé des équations (1), (8), (9) et (14), c'est–à–dire

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x})+(\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x})-\mathrm{div}\ \sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))=\mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,T), \quad \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathrm{div}\ \mathbf{u}(t,\mathbf{x})=0, & t\in(0,T), \quad \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x})=\displaystyle\sum_{j=1}^{2}\dot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,T), \quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x})=\mathbf{u}^i(t,\mathbf{x}), & t\in(0,T), \quad \mathbf{x}\in\Gamma_{\mathrm{i}},\\
\mathbf{u}(t,\mathbf{x})=0, & t\in(0,T), \quad \mathbf{x}\in\Gamma_{\mathrm{w}},\\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x})=0, & t\in(0,T), \quad \mathbf{x}\in\Gamma_{\mathrm{N}},\\
\mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}), & \qquad\qquad \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}),\\
\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}=\mathbf{M_I}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) & \\
\qquad\qquad\qquad +\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2})+\mathbf{f_s}, & t\in(0,T),\\
\theta_1(0)=\theta_{1,0}, \qquad \theta_2(0)=\theta_{2,0}, & \\
\dot{\theta}_1(0)=\omega_{1,0}, \qquad \dot{\theta}_2(0)=\omega_{2,0}. &
\end{cases}
\tag{15}
$$

Remarquons tout de suite que la modélisation que nous venons de proposer pour la structure de la Figure 2b dépendant de deux paramètres peut être étendue à des structures dépendant d'un plus grand nombre de paramètres. Par exemple, on peut considérer un solide rigide qui dépend de trois paramètres (la translation selon l'axe des abscisses, la translation selon l'axe des ordonnées et la rotation autour de son centre d'inertie). On peut alors montrer que la modélisation précédente correspond aux équations de Newton.

Plan du mémoire

Nous nous intéressons au modèle (15) ci–dessus. Nous menons une étude complète du système que nous avons introduit allant de l'étude du problème de Cauchy à la simulation numérique en passant par l'étude de la stabilisation du système continu. Ce mémoire se compose de trois chapitres décrits ci–dessous.

— Dans le premier chapitre, nous détaillons davantage la modélisation utilisée. Nous prouvons le caractère bien posé de ces équations notamment l'existence de solutions fortes en temps petits.

— Dans le second chapitre, nous nous intéressons à la stabilisation de ce problème autour d'un état stationnaire, lorsque le contrôle considéré agit sur l'équation de la structure. Nous prouvons qu'en prenant un contrôle de la forme d'une commande par retour d'état (feedback par la suite), le système en boucle fermée est stable pour des perturbations suffisamment petites. De plus, la décroissance au cours du temps de l'écart entre la solution et l'état stationnaire est exponentielle avec un taux que l'on peut choisir arbitrairement.

— Dans le dernier chapitre, nous nous intéressons à la simulation numérique en boucles ouverte et fermée du système introduit précédemment et donc, entre autres, au calcul effectif du contrôle. Après avoir prouvé la stabilité du système discrétisé en boucle fermée, nous présentons la méthode adoptée pour calculer à chaque pas de temps un contrôle correspondant à celui introduit dans le Chapitre 2. L'évolution du système d'interaction fluide–structure est simulée en utilisant une méthode de type domaine fictif sur un maillage fixe indépendant du temps et de l'état de la structure. La méthode de domaine fictif que l'on a choisie s'appuie sur des éléments finis coupés. Cette base d'éléments finis dépend de l'état de la structure et permet de rendre compte de l'évolution du domaine fluide. Le couplage entre le fluide et la structure est traité de manière partitionnée : les systèmes fluide et structure sont résolus séparément l'un après l'autre. Pour assurer le bon comportement du schéma numérique, on ajoute un terme de stabilisation localisé sur l'interface entre le fluide et la structure. Cette méthode est le prix à payer pour permettre l'utilisation d'un maillage fixe. Nous présentons des simulations numériques permettant de mettre en valeur l'efficacité du contrôle ainsi déterminé. À notre connaissance, l'utilisation des éléments finis coupés avec un contrôle est originale.

0.1 Modélisation du problème et existence de solutions fortes

0.1.1 Présentation

On s'intéresse au problème introduit dans le préambule. Dans un premier temps, nous détaillons dans ce chapitre la modélisation que nous avons présentée. Nous prouvons ensuite le caractère bien posé du problème (15). Étant donné que nous avons déjà présenté la modélisation dans le préambule, nous ne traiterons dans cette section d'introduction que les points clés concernant le caractère bien posé du problème.

0.1.2 Résultats antérieurs

De nombreux travaux sur l'existence de solutions pour les problèmes d'interaction fluide–structure considèrent un solide rigide immergé dans un fluide incompressible [147, 148, 31, 33, 88, 140, 37] ou compressible [29, 108, 105]. L'étude de structures plus complexes a été menée par exemple dans [133] où les auteurs ont étudié le problème d'une plaque plongée dans un fluide incompressible ou dans [102, 116, 107, 23] où l'interaction entre une poutre 1D et un fluide 2D a été traitée.

L'interaction entre un fluide compressible et une structure élastique a été étudiée dans [32, 30] et le cas d'une structure élastique déformable avec un fluide incompressible dans [28, 122].

Des structures déformables dont le changement de forme est donné et n'est pas calculé à partir d'une équation de structure ont été étudiées par exemple pour modéliser la nage de poissons [50, 51, 138, 106, 117].

Le cas d'une structure déformable dépendant d'un nombre fini de degrés de liberté peut être trouvé dans [34], où le modèle de la structure est une approximation des équations de l'élasticité linéaire dépendant seulement d'un nombre fini de degrés de liberté. Cependant, nous n'avons pas trouvé d'étude de structures dépendant intrinsèquement d'un nombre fini de degrés de liberté avec une dynamique libre, à part le cas des solides rigides. En cela, la modélisation que nous proposons semble originale.

Notons que les conditions mixtes sur le bord du domaine fluide imposent un cadre fonctionnel qui a été étudié dans [111] et que nous exposerons plus loin.

0.1.3 Les conditions d'entrée et la configuration de référence

Dans la suite, nous considérons que les déformations de la structure restent dans un domaine admissible : $\forall t \in [0,T],\ (\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta$, où $\mathbb{D}_\Theta$, le domaine admissible pour les paramètres de la structure, est un ouvert borné de $\mathbb{R}^2$. Pour que le problème soit bien posé, il faut que la donnée d'entrée du fluide vérifie certaines conditions de compatibilité. Plus précisément, on suppose que

$$\mathbf{u}^i \in \mathbf{U}^i = \left\{ \begin{array}{ll} \mathbf{u}^i \in \mathbf{H}^{3/2}(\Gamma_{\rm i}) \ \text{ avec } \ \mathbf{u}^i_{|\partial\Gamma_{\rm i}} = 0, & \displaystyle\int_0^{1/4} \frac{|\partial_{y_2} u_2^i(y_2)|^2}{y_2}\, \mathrm{d}y_2 < +\infty, \\ & \displaystyle\int_{3/4}^{1} \frac{|\partial_{y_2} u_2^i(y_2)|^2}{1-y_2}\, \mathrm{d}y_2 < +\infty \end{array} \right\}.$$

Pour pouvoir étudier le comportement de $\mathbf{u}$ et p dans un cadre fonctionnel plus simple, on ramène l'étude dans un domaine de référence, $\mathscr{F}_0 = \mathscr{F}(\theta_{1,0}, \theta_{2,0})$ dans notre cas. Pour cela, en suivant [34], on construit un difféomorphisme $\mathbf{\Phi^0}$ qui est une extension de $\mathbf{X}$ dans le domaine fluide. On liste ci–dessous les propriétés choisies pour $\mathbf{\Phi^0}$,

$$\begin{array}{ll} \forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, & \mathbf{\Phi^0}(\theta_1,\theta_2,S(\theta_{1,0},\theta_{2,0})) = S(\theta_1,\theta_2), \\ \forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, & \forall \mathbf{y} \in \Omega,\ \ \mathrm{d}(\mathbf{y},\partial\Omega) < \varepsilon, \quad \mathbf{\Phi^0}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y}, \\ \text{et} \quad \forall \mathbf{y} \in \Omega, & \mathbf{\Phi^0}(\theta_{1,0},\theta_{2,0},\mathbf{y}) = \mathbf{y}, \end{array} \tag{16}$$

où ε est pris suffisamment petit pour avoir

$$\forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \mathrm{d}(S(\theta_1,\theta_2),\partial\Omega) > 2\varepsilon.$$

0.1.4 Existence de solutions fortes

Le résultat principal du premier chapitre est l'existence de solutions fortes en temps petits du système d'équations (15). Plus exactement, nous prouvons le théorème suivant

Théorème 0.1.1 (Théorème 1.1.5 du Chapitre 1)**.** *Soient* $T_0 > 0$, $\mathbf{u}^i \in \mathrm{H}^1(0, T_0; \mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ *et* $(\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbb{D}_\Theta \times \mathbb{R}^2$ *satisfaisant les conditions de compatibilité*

$$\begin{cases} \operatorname{div} \mathbf{u}_0 = 0 & \text{dans } \mathscr{F}_0 = \mathscr{F}(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0(.) = \displaystyle\sum_{j=1}^{2} \omega_{j,0} \partial_{\theta_j} \mathbf{X}(\theta_{1,0}, \theta_{2,0}, \mathbf{Y}(\theta_{1,0}, \theta_{2,0}, .)) & \text{sur } \partial S_0 = \partial S(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0 = \mathbf{u}^i(0, .) & \text{sur } \Gamma_{\mathrm{i}}, \\ \mathbf{u}_0 = 0 & \text{sur } \Gamma_{\mathrm{w}}. \end{cases} \tag{17}$$

Soient $\mathbf{f}_{\mathscr{F}} \in \mathrm{L}^2(0, T_0; \mathbf{W}^{1,\infty}(\Omega))$ *et* $\mathbf{f_s} \in \mathrm{L}^2(0, T_0; \mathbb{R}^2)$. *Alors il existe un temps final* $T \in (0, T_0]$ *et une constante* $C > 0$ *tels que le problème* (15) *admet une unique solution* $(\mathbf{u}, p, \theta_1, \theta_2)$ *avec la régularité suivante*

$$\begin{aligned} &(\theta_1, \theta_2) \in \mathrm{H}^2(0, T; \mathbb{D}_\Theta), \\ &\mathbf{u}(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y})) \in \mathrm{L}^2(0, T; \mathbf{H}^{3/2}(\mathscr{F}_0)) \cap \mathscr{C}^0([0, T]; \mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{H}^1(0, T; \mathbf{L}^2(\mathscr{F}_0)), \\ &p(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y})) \in \mathrm{L}^2(0, T; \mathrm{H}^{1/2}(\mathscr{F}_0)). \end{aligned}$$

De plus, cette solution satisfait l'estimée suivante

$$\begin{aligned} &\|\mathbf{u}(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y}))\|_{\mathrm{L}^2(0,T;\mathbf{H}^{3/2}(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} \\ &\quad + \|p(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y}))\|_{\mathrm{L}^2(0,T;\mathrm{H}^{1/2}(\mathscr{F}_0))} + \|(\theta_1, \theta_2)\|_{\mathrm{H}^2(0,T;\mathbb{D}_\Theta)} \\ &\quad \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_{\mathscr{F}}\|_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} \\ &\qquad + \|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{H}^{3/2}(\Gamma_{\mathrm{i}}))} + \|\mathbf{f_s}\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^2)}). \end{aligned}$$

Remarque 0.1.2. Le résultat d'existence établi ci–dessus est en fait prouvé dans un cadre fonctionnel qui fait intervenir les espaces de Sobolev $\mathrm{L}^2(0, T; \mathbf{H}^2_\beta(\mathscr{F}_0))$ pour la vitesse et $\mathrm{L}^2(0, T; \mathrm{H}^1_\beta(\mathscr{F}_0))$ pour la pression. Ils correspondent aux espaces de Sobolev classiques auxquels on a ajouté un poids près des coins du domaine pour prendre en compte d'éventuelles singularités. On notera qu'on a en particulier les inclusions $\mathbf{H}^2_\beta(\mathscr{F}_0) \subset \mathbf{H}^{3/2}(\mathscr{F}_0)$ et $\mathrm{H}^1_\beta(\mathscr{F}_0) \subset \mathrm{H}^{1/2}(\mathscr{F}_0)$. Pour plus d'informations, le lecteur peut se reporter au Chapitre 1 ou à [111].

0.1.5 Plan de la preuve

La preuve du Théorème 0.1.1 que nous proposons contient les étapes suivantes :
- réécriture du système (15) dans le domaine fixe $\mathscr{F}_0$,
- étude de l'existence de solutions au système linéarisé en domaine fixe. Cette étape utilise l'écriture du système linéaire sous la forme d'un semi–groupe,
- existence de solutions au problème non linéaire par un argument de point fixe.

L'argument de point fixe s'appuie sur les résultats obtenus pour le système linéarisé. Il est donc nécessaire que les constantes dans les estimées obtenues sur le système linéaire soient indépendantes du temps. Il s'agit là de la principale difficulté de cette preuve. Le plan suivi est le même que dans [34].

Pour simplifier les notations, nous menons cette preuve avec $\theta_{1,0} = \theta_{2,0} = 0$ *et nous supposons que* $(0, 0) \in \mathbb{D}_\Theta$. *Ce choix ne restreint pas la généralité de notre propos puisque nous pouvons retrouver le cas général à l'aide d'un changement de variables.*

0.1.5.1 Réécriture du problème en domaine fixe

Une difficulté classique rencontrée dans le cadre des problèmes d'interaction fluide–structure est que le domaine fluide dépend de l'état de la structure et peut donc changer au cours du temps. La méthode la plus utilisée pour surmonter cette difficulté, celle que nous avons choisie pour cette étude, est de définir un difféomorphisme qui relie le domaine fluide à l'instant t avec un domaine fluide de référence et d'effectuer un changement de variables avec ce difféomorphisme. Nous utilisons le changement de variables suivant

$$\forall \mathbf{y} \in \mathscr{F}_0, \quad \forall t \in (0,T), \qquad \left\{ \begin{array}{l} \tilde{\mathbf{u}}(t,\mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1(t),\theta_2(t),\mathbf{y}))^T \mathbf{u}(t, \mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y})), \\ \tilde{p}(t,\mathbf{y}) = p(t, \mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y})), \end{array} \right. \tag{18}$$

où $\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1(t),\theta_2(t),\mathbf{y}) = \nabla \mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y})$ est la matrice jacobienne de $\mathbf{\Phi^0}$. Ce changement de variables a été choisi pour assurer à $\tilde{\mathbf{u}}$ une divergence nulle. Après avoir déterminé le système d'équations vérifié par le nouvel état $(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2)$, nous gardons les termes linéaires et nous regroupons tous les termes non linéaires dans les termes sources et les termes frontière $\mathbf{f}$, $\mathbf{g}$ et $\mathbf{s}$. Nous obtenons le système suivant

$$\left\{ \begin{array}{ll} \dfrac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu \Delta \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} & \text{dans } (0,T)\times\mathscr{F}_0, \\ \mathrm{div}\, \tilde{\mathbf{u}} = 0 & \text{dans } (0,T)\times\mathscr{F}_0, \\ \tilde{\mathbf{u}} = \dot{\theta}_1 \partial_{\theta_1}\mathbf{\Phi^0}(0,0,.) + \dot{\theta}_2 \partial_{\theta_2}\mathbf{\Phi^0}(0,0,.) + \mathbf{g} & \text{sur } (0,T)\times\partial S_0, \\ \tilde{\mathbf{u}} = \mathbf{u}^i & \text{sur } (0,T)\times\Gamma_{\mathrm{i}}, \\ \tilde{\mathbf{u}} = 0 & \text{sur } (0,T)\times\Gamma_{\mathrm{w}}, \\ \sigma_F(\tilde{\mathbf{u}},\tilde{p})\mathbf{n} = 0 & \text{sur } (0,T)\times\Gamma_{\mathrm{N}}, \\ \tilde{\mathbf{u}}(0,.) = \mathbf{u}_0(.) & \text{dans } \mathscr{F}_0, \\ \mathcal{M}_{0,0}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0 \cdot \partial_{\theta_1}\mathbf{\Phi^0}(0,0,\gamma_y)\, \mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0 \cdot \partial_{\theta_2}\mathbf{\Phi^0}(0,0,\gamma_y)\, \mathrm{d}\gamma_y \end{pmatrix} + \mathbf{s} & \text{sur } (0,T), \\ \theta_1(0) = 0, \qquad \theta_2(0) = 0, & \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}, & \end{array} \right. \tag{19}$$

où $\mathbf{n}_0$ est la normale sortante à $\mathscr{F}_0$ sur ∂S_0 et les termes sources sont donnés par

$$\left\{ \begin{array}{l} \mathbf{f} = \mathbf{F}(\theta_1,\theta_2,\tilde{\mathbf{u}},\tilde{p}) + \mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y})), \\ \mathbf{g} = \mathbf{G}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2), \\ \mathbf{s} = \mathbf{S}(\theta_1,\theta_2,\tilde{\mathbf{u}},\tilde{p}) + \mathbf{f_s}. \end{array} \right. \tag{20}$$

Les termes non linéaires $\mathbf{F}$, $\mathbf{G}$ et $\mathbf{S}$ sont définis dans le Chapitre 1 en (1.66).

0.1.5.2 Étude du semi–groupe associé au problème linéarisé

Pour étudier le problème (19) avec $\mathbf{f} = 0$, $\mathbf{g} = 0$ et $\mathbf{s} = 0$, on introduit les espaces fonctionnels suivants qui sont adaptés à notre problème :

$$\mathbb{H}_0 = \left\{ \begin{array}{lll} (\tilde{\mathbf{u}},\theta_1,\theta_2,\omega_1,\omega_2) \in \mathbf{L}^2(\mathscr{F}_0)\times\mathbb{R}^4 & \text{avec} & \mathrm{div}\,\tilde{\mathbf{u}} = 0 \text{ dans } \mathscr{F}_0, \quad \tilde{\mathbf{u}}\cdot\mathbf{n} = 0 \text{ sur } \Gamma_{\mathrm{D}}, \\ & & \tilde{\mathbf{u}}\cdot\mathbf{n}_0 = \sum_j \omega_j \partial_{\theta_j}\mathbf{\Phi^0}(0,0,.)\cdot\mathbf{n}_0 \text{ sur } \partial S_0 \end{array} \right\},$$

et

$$\mathbb{V}_0 = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{H}^1(\mathscr{F}_0) \times \mathbb{R}^4 \ \text{ avec } & \text{div } \tilde{\mathbf{u}} = 0 \text{ dans } \mathscr{F}_0, \quad \tilde{\mathbf{u}} = 0 \text{ sur } \Gamma_{\mathrm{D}}, \\ & \tilde{\mathbf{u}} = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi^0}(0,0,.) \text{ sur } \partial S_0 \end{array} \right\}.$$

Nous définissons ensuite un opérateur A_0 sur $\mathbb{H}_0$ par

$$D(A_0) = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}_0 \ \text{ avec } & \exists \tilde{p} \in \mathrm{L}^2(\mathscr{F}_0) \text{ tel que} \\ & \text{div } \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \in \mathbf{L}^2(\mathscr{F}_0) \text{ et } \sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} = 0 \text{ sur } \Gamma_{\mathrm{N}} \end{array} \right\},$$

et

$$A_0 \begin{pmatrix} \tilde{\mathbf{u}} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}_0} \begin{pmatrix} \text{div } \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi^0}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \right)_{j=1,2} \end{pmatrix},$$

où $\Pi_{\mathbb{H}_0}$ est la projection orthogonale de $\mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ sur $\mathbb{H}_0$.

Nous prouvons que A_0 engendre un semi–groupe analytique sur $\mathbb{H}_0$, ce qui implique que, pour toute donnée initiale $\mathbf{z}_0$ dans $\mathbb{V}_0$ et tout terme source $\mathbf{F}$ dans $\mathrm{L}^2(0,T;\mathbb{H}_0)$, le problème

$$\left\{ \begin{array}{l} \mathbf{z}'(t) = A_0 \mathbf{z}(t) + \mathbf{F}(t), \quad t \in (0,T), \\ \mathbf{z}(0) = \mathbf{z}_0, \end{array} \right. \tag{21}$$

admet une unique solution $\mathbf{z} \in \mathrm{H}^1(0,T;\mathbb{H}_0) \cap \mathrm{L}^2(0,T;D(A_0))$.

0.1.5.3 Étude du problème linéarisé

On s'intéresse à l'existence de solutions au problème linéarisé (19) avec des termes source donnés dans

$$\begin{array}{l} \mathbf{f} \in \mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0)), \\ \mathbf{g} \in \mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial S_0)), \\ \mathbf{s} \in \mathrm{L}^2(0,T;\mathbb{R}^2). \end{array}$$

De telles solutions sont cherchées sous la forme $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) = (\hat{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) + (\overline{\mathbf{u}}, 0, 0, 0)$ où $\overline{\mathbf{u}}$ est un relèvement des données frontières $\mathbf{u}^i$ et $\mathbf{g}$, c'est–à–dire

$$\left\{ \begin{array}{ll} \text{div } \overline{\mathbf{u}} = 0 & \text{dans } (0,T) \times \mathscr{F}_0, \\ \overline{\mathbf{u}} = \mathbf{g} & \text{sur } (0,T) \times \partial S_0, \\ \overline{\mathbf{u}} = \mathbf{u}^i & \text{sur } (0,T) \times \Gamma_{\mathrm{i}}, \\ \overline{\mathbf{u}} = 0 & \text{sur } (0,T) \times \Gamma_{\mathrm{w}}, \\ (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T)\mathbf{n} = 0 & \text{sur } (0,T) \times \Gamma_{\mathrm{N}}. \end{array} \right.$$

L'existence d'un tel relèvement $\overline{\mathbf{u}}$ est établie dans le Lemme 1.2.8.

On peut alors montrer que $\mathbf{z} = (\hat{\mathbf{u}}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ est solution du problème (21) pour un $\mathbf{F}$ bien choisi. L'existence d'un tel $\mathbf{z}$ a été prouvée dans la Section 0.1.5.2. Cela établit l'existence d'une solution $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ au problème (19) dans $\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$, ces espaces étant définis par

$$\mathbb{U}_T = \mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0)),$$

$$\mathbb{P}_T = \mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0)),$$
$$\Theta_T = \mathrm{H}^2(0,T;\mathbb{D}_\Theta),$$

où $\mathbf{H}^2_\beta(\mathscr{F}_0)$ et $\mathrm{H}^1_\beta(\mathscr{F}_0)$ sont des espaces de Sobolev auxquels on a ajouté des poids pour considérer des solutions présentant des singularités près des coins du domaine. Le lecteur trouvera plus d'informations en Section 1.1.2.1.

0.1.5.4 Traitement des termes non linéaires par un argument de point fixe

La dernière étape de la preuve du Théorème 0.1.1 consiste à traiter localement les termes non linéaires grâce à un argument de point fixe. On définit l'espace

$$\mathbb{N}_T = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T \ \text{ avec } & \forall t \in (0,T), \quad (\theta_1,\theta_2)(t) \in \mathbb{D}_\Theta, \\ & \theta_1(0)=0, \quad \theta_2(0)=0, \quad \dot{\theta}_1(0)=\omega_{1,0}, \quad \dot{\theta}_2(0)=\omega_{2,0} \end{array} \right\},$$

et l'application

$$\Lambda^T : \mathbb{N}_T \to \mathbb{N}_T$$

par $\Lambda^T(\overline{\mathbf{u}},\overline{p},\overline{\theta_1},\overline{\theta_2}) = (\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2)$ où $(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2)$ est la solution du système (19) avec

$$\left\{ \begin{array}{l} \mathbf{f} = \mathbf{F}(\overline{\theta_1},\overline{\theta_2},\overline{\mathbf{u}},\overline{p}) + \mathbf{f}_{\mathscr{F}}(t,\mathbf{\Phi^0}(\overline{\theta_1},\overline{\theta_2},\mathbf{y})), \\ \mathbf{g} = \mathbf{G}(\overline{\theta_1},\overline{\theta_2},\dot{\overline{\theta_1}},\dot{\overline{\theta_2}}), \\ \mathbf{s} = \mathbf{S}(\overline{\theta_1},\overline{\theta_2},\overline{\mathbf{u}},\overline{p}). \end{array} \right.$$

Nous montrons, grâce à des estimations sur les termes non linéaires $\mathbf{F}$, $\mathbf{G}$, $\mathbf{S}$ que, pour R suffisamment grand et un temps final T suffisamment petit, l'application Λ^T est une contraction sur la boule

$$\mathbb{B}_R(T) = \{(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2) \in \mathbb{N}_T \ \text{ avec } \ \|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|(\theta_1,\theta_2)\|_{\Theta_T} \leq R\}.$$

Le point clé de cette démonstration est que les estimées de la partie linéaire ont des constantes indépendantes du temps T. Ceci permet alors de montrer que Λ^T est une contraction pour T assez petit.

Ainsi, d'après le théorème du point fixe de Picard, pour des données initiales vérifiant les conditions (17), il existe un temps final T tel que le système (19)–(20) admet une unique solution dans $\mathbb{B}_R(T)$. Un changement de variable permet alors de conclure la preuve du Théorème 0.1.1.

0.2 Stabilisation locale du système fluide–structure

0.2.1 Présentation

Dans cette section, nous construisons un opérateur de feedback qui permet de stabiliser le système du chapitre précédent autour d'un état d'équilibre a priori instable, avec un contrôle agissant sur la structure. Le système d'équations que nous considérons est le suivant :

$$\begin{cases}\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x})+(\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x})-\mathrm{div}\ \sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))=\mathbf{f}_{\mathscr{F}}(\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\ \mathrm{div}\ \mathbf{u}(t,\mathbf{x})=0, & t\in(0,\infty),\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\ \mathbf{u}(t,\mathbf{x})=\displaystyle\sum_{j=1}^{2}\dot\theta_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,\infty),\ \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)),\\ \mathbf{u}(t,\mathbf{x})=\mathbf{u}^i(\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\mathrm{i}},\\ \mathbf{u}(t,\mathbf{x})=0, & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\mathrm{w}},\\ \sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x})=0, & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\mathrm{N}},\\ \mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}),\\ \mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix}=\mathbf{M}_{\mathbf{I}}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) & \\ \qquad\qquad +\mathbf{M}_{\mathbf{A}}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2})+\mathbf{f_s}+\mathbf{h}, & t\in(0,\infty),\\ \theta_1(0)=\theta_{1,0},\quad \theta_2(0)=\theta_{2,0}, & \\ \dot\theta_1(0)=\omega_{1,0},\quad \dot\theta_2(0)=\omega_{2,0}, & \end{cases}\tag{22}$$

où $\mathcal{M}_{\theta_1,\theta_2}$, $\mathbf{M_A}$, $\mathbf{M_I}$ sont définis par (10)–(13), $(\mathbf{f}_{\mathscr{F}},\mathbf{u}^i,\mathbf{f_s})\in\mathbf{W}^{1,\infty}(\Omega)\times\mathbf{U}^i\times\mathbb{R}^2$ sont des données stationnaires et $\mathbf{h}$ est le contrôle.

Nous considérons une solution stationnaire $(\mathbf{w},p_{\mathbf{w}},0,0)\in\mathbf{H}^{3/2}(\mathscr{F}(0,0))\times\mathrm{H}^{1/2}(\mathscr{F}(0,0))\times\mathbb{R}^2$, satisfaisant le système suivant

$$\begin{cases}(\mathbf{w}\cdot\nabla)\mathbf{w}-\mathrm{div}\ \sigma_F(\mathbf{w},p_{\mathbf{w}})=\mathbf{f}_{\mathscr{F}} & \text{dans } \mathscr{F}(0,0),\\ \mathrm{div}\ \mathbf{w}=0 & \text{dans } \mathscr{F}(0,0),\\ \mathbf{w}=0 & \text{sur } \Gamma_{\mathrm{w}}\cup\partial S(0,0),\\ \mathbf{w}=\mathbf{u}^i & \text{sur } \Gamma_{\mathrm{i}},\\ \sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n}=0 & \text{sur } \Gamma_{\mathrm{N}},\\ \left(\displaystyle\int_{\partial S(0,0)}\sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n}_s\cdot\partial_{\theta_j}\mathbf{X}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2}=\mathbf{f_s}, & \end{cases}\tag{23}$$

où $\mathbf{n}_s$ est la normale unitaire sortante à $\mathscr{F}_s=\mathscr{F}(0,0)$ sur $\partial S(0,0)$. La configuration cible est $S_s=S(0,0)$ pour le domaine de la structure et $\mathscr{F}_s=\mathscr{F}(0,0)$ pour le domaine du fluide.

Le but du Chapitre 2 est de construire un opérateur de feedback $\mathcal{K}_\delta$ tel que $(\mathbf{w},p_{\mathbf{w}},0,0)$ soit un équilibre stable du système (22) avec un contrôle $\mathbf{h}$ déterminé à partir de $\mathcal{K}_\delta$.

Pour cela, on s'intéresse à l'évolution de l'écart entre la solution écrite en domaine fixe et la solution stationnaire

$$\forall\mathbf{y}\in\mathscr{F}_s,\ \forall t\in(0,\infty),\ \begin{cases}\mathbf{v}(t,\mathbf{y})=\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1(t),\theta_2(t),\mathbf{y}))^T\mathbf{u}(t,\mathbf{\Phi^S}(\theta_1(t),\theta_2(t),\mathbf{y}))-\mathbf{w}(\mathbf{y}),\\ q(t,\mathbf{y})=p(t,\mathbf{\Phi^S}(\theta_1(t),\theta_2(t),\mathbf{y}))-p_{\mathbf{w}}(\mathbf{y}).\end{cases}\tag{24}$$

Le quadruplet $(\mathbf{v}, q, \theta_1, \theta_2)$ est alors solution du système

$$
\left\{
\begin{array}{ll}
\dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{w}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{w} - \mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y}) - \operatorname{div}\sigma_F(\mathbf{v},q) = \mathbf{f} & \text{dans } (0,\infty)\times\mathscr{F}_s,\\
\operatorname{div}\mathbf{v} = 0 & \text{dans } (0,\infty)\times\mathscr{F}_s,\\
\mathbf{v} = \dot\theta_1\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y) + \dot\theta_2\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y) + \mathbf{g} & \text{sur } (0,\infty)\times\partial S_s,\\
\mathbf{v} = 0 & \text{sur } (0,\infty)\times\Gamma_{\mathrm{D}},\\
\sigma_F(\mathbf{v},q)\mathbf{n} = 0 & \text{sur } (0,\infty)\times\Gamma_{\mathrm{N}},\\
\mathbf{v}(0,\mathbf{y}) = \mathbf{v}_0(\mathbf{y}) = \operatorname{cof}(\mathscr{J}_{\mathbf{\Phi^S}}(\theta_{1,0},\theta_{2,0},\mathbf{y}))^T\mathbf{u}_0(\mathbf{\Phi^S}(\theta_{1,0},\theta_{2,0},\mathbf{y})) - \mathbf{w}(\mathbf{y}) & \qquad\qquad \text{dans } \mathscr{F}_s,\\
\mathcal{M}_{0,0}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v},q)\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)\,\mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v},q)\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)\,\mathrm{d}\gamma_y \end{pmatrix} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\mathbf{L_S}(\theta_1,\theta_2) + \mathbf{s} + \mathbf{h} & \text{sur } (0,\infty),\\
\theta_1(0) = \theta_{1,0}, \qquad \theta_2(0) = \theta_{2,0}, & \\
\dot\theta_1(0) = \omega_{1,0}, \qquad \dot\theta_2(0) = \omega_{2,0}, &
\end{array}
\right.
\tag{25}
$$

où les termes $\mathbf{L_F}$ et $\mathbf{L_S}$ sont linéaires en θ_1, θ_2, $\dot\theta_1$ et $\dot\theta_2$

$$
\left\{
\begin{array}{l}
\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y}) = \mathbf{L}_1(\mathbf{y})\theta_1 + \mathbf{L}_2(\mathbf{y})\theta_2 + \mathbf{L}_3(\mathbf{y})\dot\theta_1 + \mathbf{L}_4(\mathbf{y})\dot\theta_2,\\
\mathbf{L_S}(\theta_1,\theta_2) = \mathbf{L}_5\theta_1 + \mathbf{L}_6\theta_2,
\end{array}
\right.
$$

et où les coefficients $\mathbf{L}_1$–$\mathbf{L}_6$ sont donnés en (B.1)–(B.6). De plus, on a

$$
\left\{
\begin{array}{l}
\mathbf{f} = \mathbf{f}^{NL}(\theta_1,\theta_2,\mathbf{v},q),\\
\mathbf{g} = \mathbf{g}^{NL}(\theta_1,\theta_2),\\
\mathbf{s} = \mathbf{s}^{NL}(\theta_1,\theta_2,\mathbf{v},q),
\end{array}
\right.
\tag{26}
$$

où les termes non linéaires $\mathbf{f}^{NL}$, $\mathbf{g}^{NL}$ et $\mathbf{s}^{NL}$ sont donnés en (2.67).

Stabiliser le système (22) autour de l'état stationnaire $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$ revient à stabiliser (25)–(26) autour de l'état nul $(0,0,0,0)$.

0.2.2 Résultats antérieurs

La stabilisation d'équations représentant un fluide a été étudiée par exemple dans [67, 20, 18, 16, 77, 78, 115]. Une stratégie largement utilisée pour obtenir ce résultat de stabilisation consiste à calculer la valeur du contrôle à partir de l'état du système, on construit alors un opérateur de feedback [68, 58, 69, 70, 15, 19, 17, 98, 149, 132]. Cet opérateur de feedback peut être déterminé à partir de la solution d'une équation de Riccati [118, 128, 129]. Pour pallier d'éventuelles conditions de compatibilité, on peut calculer le contrôle comme étant la solution d'une équation prenant en compte l'état du système [7, 8].

La stabilisation d'un système d'interaction fluide–structure a été étudiée par exemple dans [143, 6, 48, 148, 11, 50, 51, 9, 116, 131]. On peut, une fois de plus, déterminer l'opérateur de feedback que nous utilisons à partir de la solution d'une équation de Riccati [9, 116, 131].

En particulier, dans [9], les auteurs stabilisent par feedback la solution d'un problème d'interaction fluide–structure mettant en jeu un solide rigide couplé aux équations de Navier–Stokes. La structure est donc donnée, comme dans notre cas, par un nombre fini de degrés de liberté. Nous nous inspirons donc de cette étude par la suite pour construire notre preuve. Le caractère déformable de la structure génère un terme non linéaire à l'interface entre le fluide et la structure.

Notons également que, contrairement à [9], le contrôle agit comme une force sur l'équation de la structure au lieu d'agir en tant que donnée de Dirichlet sur le bord du domaine fluide.

0.2.3 Propriété de continuation unique et résultat de stabilisation

Comme dans le chapitre précédent, nous introduisons un difféomorphisme $\mathbf{\Phi^S}(\theta_1, \theta_2, .)$: $\Omega \to \Omega$ qui lie respectivement les domaines de référence S_s et $\mathscr{F}_s$ aux domaines à l'instant t, $S(\theta_1, \theta_2)$ et $\mathscr{F}(\theta_1, \theta_2)$. Ce difféomorphisme présente des propriétés similaires à celles de $\mathbf{\Phi^0}$:

$$\begin{aligned} &\forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \mathbf{\Phi^S}(\theta_1,\theta_2,S(0,0)) = S(\theta_1,\theta_2), \\ &\forall(\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \forall \mathbf{y} \in \Omega, \ \mathrm{d}(\mathbf{y}, \partial\Omega) < \varepsilon, \quad \mathbf{\Phi^S}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y}, \\ &\qquad\qquad\qquad\qquad\qquad \text{et} \quad \forall \mathbf{y} \in \Omega, \quad \mathbf{\Phi^S}(0,0,\mathbf{y}) = \mathbf{y}. \end{aligned} \tag{27}$$

Pour prouver notre résultat, nous devons d'abord admettre le résultat de continuation unique suivant qui dépend d'un taux de stabilisation $\delta > 0$ arbitrairement choisi. Notons que des termes $\mathbf{L}_1$–$\mathbf{L}_6$ apparaissent dans cette hypothèse, ces termes sont des linéarisations de termes non linéaires, ils sont définis dans l'Annexe B en (B.1)–(B.6).

Hypothèse $(\boldsymbol{\mathcal{H}})_\delta$ (Propriété de continuation unique)**.** Tout vecteur propre $(\mathbf{v}, q, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathrm{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ du problème adjoint associé à (25) pour la valeur propre $\overline{\lambda}$ avec $\mathcal{R}e(\overline{\lambda}) \geq -\delta$, c'est–à–dire toute solution de

$$\begin{cases} \operatorname{div} \sigma_F(\mathbf{v}, q) - (\nabla \mathbf{w})^T \mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{v} = \overline{\lambda}\mathbf{v} & \text{dans } \mathscr{F}_s, \\ \operatorname{div} \mathbf{v} = 0 & \text{dans } \mathscr{F}_s, \\ \mathbf{v} = \omega_1 \partial_{\theta_1} \mathbf{\Phi^S}(0,0,.) + \omega_2 \partial_{\theta_2} \mathbf{\Phi^S}(0,0,.) & \text{sur } \partial S_s, \\ \mathbf{v} = 0 & \text{sur } \Gamma_\mathrm{D}, \\ \sigma_F(\mathbf{v}, q)\mathbf{n} + (\mathbf{w} \cdot \mathbf{n})\mathbf{v} = 0 & \text{sur } \Gamma_\mathrm{N}, \\ \displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_1(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \\ \mathbf{L}_2(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} + \begin{pmatrix} \mathbf{L}_5 \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ \mathbf{L}_6 \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \end{pmatrix} = \overline{\lambda} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, & \\ \displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_3(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \\ \mathbf{L}_4(\mathbf{y}) \cdot \mathbf{v}(\mathbf{y}) \end{pmatrix} \mathrm{d}\mathbf{y} - \int_{\partial S_s} \begin{pmatrix} \sigma_F(\mathbf{v}, q)\mathbf{n}_s(\gamma_y) \cdot \partial_{\theta_1} \mathbf{\Phi^S}(0,0,\gamma_y) \\ \sigma_F(\mathbf{v}, q)\mathbf{n}_s(\gamma_y) \cdot \partial_{\theta_2} \mathbf{\Phi^S}(0,0,\gamma_y) \end{pmatrix} \mathrm{d}\gamma_y + \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \overline{\lambda} \mathcal{M}_{0,0} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, & \end{cases}$$

qui appartient au noyau de l'adjoint de l'opérateur de contrôle, c'est–à–dire qui satisfait

$$\begin{cases} \omega_1 = 0, \\ \omega_2 = 0, \end{cases}$$

est nécessairement nul, c'est–à–dire $(\mathbf{v}, q, \theta_1, \theta_2, \omega_1, \omega_2) = (0, 0, 0, 0, 0, 0)$.

Un exemple de résultat de continuation unique est disponible pour le système de Stokes avec une observation locale [61]. Ici, l'observation (ω_1, ω_2) correspond aux dérivées des paramètres de la structure. L'information que nous obtenons sur le fluide est le fait que l'intégrale de la force que celui–ci exerce sur la structure est nulle. Cette information est non locale et le résultat de continuation unique dont nous avons besoin est, à notre connaissance, indisponible dans la littérature. Même si rien ne laisse penser que cette hypothèse peut ne pas être vérifiée, prouver un tel résultat nous semble hors de portée à moyen terme.

Le résultat de continuation unique que nous admettons porte sur l'adjoint du système linéarisé (25) associé au problème non linéaire (23). Par exemple, les termes $-(\nabla \mathbf{w})^T \mathbf{v} + (\mathbf{w} \cdot \nabla)\mathbf{v}$ proviennent du terme non linéaire $\mathbf{u} \cdot \nabla \mathbf{u}$.

Remarque 0.2.1. Les coefficients $\mathbf{L}_1$–$\mathbf{L}_6$ présents dans $(\boldsymbol{\mathcal{H}})_\delta$ dépendent du choix du difféomorphisme $\mathbf{\Phi}^\mathbf{S}$. Cependant, l'hypothèse $(\boldsymbol{\mathcal{H}})_\delta$ est indépendante du choix de ce difféomorphisme du moment qu'il satisfait les propriétés (27) et est donc intrinsèque au modèle considéré. Le lecteur peut se reporter à l'Annexe C pour plus d'information.

Dans la suite, nous travaillons sous l'hypothèse $(\boldsymbol{\mathcal{H}})_\delta$. Il est à noter que nous pouvons vérifier cette propriété lors de simulations numériques sur chaque cas étudié. Le lecteur peut se reporter au Chapitre 3 où cette question est abordée.

Nous obtenons le résultat suivant.

Théorème 0.2.2 (Théorème 2.1.5 du Chapitre 2)**.** *Soit $\delta > 0$ tel que l'hypothèse $(\boldsymbol{\mathcal{H}})_\delta$ est vérifiée. Soient $\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega)$, $\mathbf{u}^i \in \mathbf{U}^i$, $\mathbf{f_s} \in \mathbb{R}^2$, et $(\mathbf{w}, p_\mathbf{w}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ satisfaisant* (23)*. Alors, il existe un opérateur de feedback $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$, $\varepsilon > 0$ et $C > 0$ tels que pour toutes données initiales $(\mathbf{u}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}(\theta_{1,0}, \theta_{2,0})) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ satisfaisant les conditions de compatibilité*

$$\begin{cases} \operatorname{div} \mathbf{u}_0 = 0 & \text{dans } \mathscr{F}(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0(.) = \displaystyle\sum_{j=1}^{2} \omega_{j,0} \partial_{\theta_j} \mathbf{X}(\theta_{1,0}, \theta_{2,0}, \mathbf{Y}(\theta_{1,0}, \theta_{2,0}, .)) & \text{sur } \partial S(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0 = \mathbf{u}^i & \text{sur } \Gamma_\mathrm{i}, \\ \mathbf{u}_0 = 0 & \text{sur } \Gamma_\mathrm{w}, \end{cases} \tag{28}$$

et la condition de petitesse

$$\|\mathbf{u}_0(\mathbf{\Phi}^\mathbf{S}(\theta_{1,0}, \theta_{2,0}, .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \leq \varepsilon,$$

la solution $(\mathbf{u}, p, \theta_1, \theta_2)$ de (22) *avec*

$$\mathbf{h}(t) = \mathcal{K}_\delta\Big(\Big[\operatorname{cof}(\mathcal{J}_{\mathbf{\Phi}^\mathbf{S}}(\theta_1(t), \theta_2(t), .))^T \mathbf{u}(t, \mathbf{\Phi}^\mathbf{S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}\Big], \theta_1(t), \theta_2(t), \dot\theta_1(t), \dot\theta_2(t)\Big), \tag{29}$$

satisfait pour tout t dans $(0, \infty)$,

$$\|\mathbf{u}(t, \mathbf{\Phi}^\mathbf{S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot\theta_1(t)| + |\dot\theta_2(t)| \leq Ce^{-\delta t}.$$

0.2.4 Plan de la preuve

La preuve du Théorème 0.2.2 est composée des étapes suivantes :

— on construit un opérateur de feedback qui stabilise le système linéarisé (22) autour de l'état nul,

— on prouve que l'opérateur de feedback construit dans la partie linéaire stabilise aussi le système non linéaire pour des perturbations suffisamment petites.

La principale difficulté de cette preuve est de trouver le bon cadre fonctionnel qui permet de ramener le problème sous contrainte (la continuité des vitesses à l'interface et la divergence nulle) à un problème de contrôle classique pour pouvoir utiliser la littérature.

0.2.4.1 Étude du semi–groupe associé au problème linéarisé en boucle ouverte

En suivant une démarche semblable à la Section 0.1.5.2, nous définissons un cadre fonctionnel adapté à notre étude. Les espaces suivants y sont similaires, seuls changent le difféomorphisme utilisé, $\mathbf{\Phi^S}$ au lieu de $\mathbf{\Phi^0}$, et le domaine fluide de référence, $\mathscr{F}_s$ au lieu de $\mathscr{F}_0$.

Nous définissons les espaces suivants

$$\mathbb{H}_S = \left\{ \begin{array}{r} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4 \ \text{ avec div } \mathbf{v} = 0 \text{ dans } \mathscr{F}_s, \quad \mathbf{v}\cdot\mathbf{n} = 0 \text{ sur } \Gamma_{\mathrm{D}} \\ \mathbf{v}\cdot\mathbf{n}_s = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi^S}(0,0,.)\cdot\mathbf{n}_s \text{ sur } \partial S_s \end{array} \right\}, \tag{30}$$

et

$$\mathbb{V}_S = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4 \ \text{ avec} & \text{div } \mathbf{v} = 0 \text{ dans } \mathscr{F}_s, \quad \mathbf{v} = 0 \text{ sur } \Gamma_{\mathrm{D}} \\ & \mathbf{v} = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi^S}(0,0,.) \text{ sur } \partial S_s \end{array} \right\}.$$

Nous définissons l'opérateur A_S sur $\mathbb{H}_S$ par

$$D(A_S) = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}_S \ \text{ avec} & \exists q \in \mathrm{L}^2(\mathscr{F}_s) \ \text{ tel que} \\ & \text{div } \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \ \text{ et } \ \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 \text{ sur } \Gamma_{\mathrm{N}} \end{array} \right\},$$

et

$$A_S \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}_S} \begin{pmatrix} \text{div } \sigma_F(\mathbf{v}, q) + \mathbf{L_F}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{v} \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi^S}(0,0,\gamma_y)\, \mathrm{d}\gamma_y \right)_{j=1,2} + \mathbf{L_S}(\theta_1, \theta_2) \right) \end{pmatrix},$$

où $\Pi_{\mathbb{H}_S}$ est la projection orthogonale de $\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ sur $\mathbb{H}_S$. On montre ensuite que l'opérateur A_S engendre un semi–groupe analytique et qu'il est à résolvante compacte. Notons que, contrairement au Chapitre 1, nous nous intéressons à la résolvante de l'opérateur pour en déduire des informations sur le spectre de A_S. Ces dernières seront utilisées pour construire un opérateur de feedback.

Notons que A_S contient des termes supplémentaires par rapport à A_0. Il s'agit des termes linéaires supplémentaires présents dans (22).

0.2.4.2 Construction d'un opérateur de feedback

L'opérateur de contrôle B est borné et est donné par

$$B\mathbf{h} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\mathbf{h} \end{pmatrix},$$

où $\mathcal{M}_{0,0}$ est donné en (10).

Nous considérons donc le système

$$\left\{ \begin{array}{l} \mathbf{z}'(t) = (A_S + \delta \mathrm{I})\mathbf{z}(t) + B\mathbf{h}(t), \quad t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0. \end{array} \right. \tag{31}$$

Le but est alors de construire un opérateur de feedback $\mathcal{K}_\delta$ tel que la solution du problème (31) avec $\mathbf{h}(t) = \mathcal{K}_\delta \mathbf{z}(t)$ décroît au cours du temps.

Pour cela, on définit J_u l'ensemble des indices $j \in \mathbb{N}$ tels que la valeur propre λ_j de A_S vérifie $\mathcal{R}e(\lambda_j) \geq -\delta$. On définit également $\mathrm{G}(\lambda_j)$ l'espace propre généralisé de A_S associé à la valeur propre λ_j et

$$\mathbb{Z}_u = \bigoplus_{j \in J_u} \mathrm{G}(\lambda_j),$$

l'espace engendré par les vecteurs propres généralisés instables de $A_S + \delta\mathrm{I}$. On peut alors utiliser la décomposition

$$\mathbb{H}_S = \mathbb{Z}_u \bigoplus \mathbb{Z}_s,$$

où $\mathbb{Z}_s$ peut être défini de manière similaire à $\mathbb{Z}_u$ comme l'espace engendré par les vecteurs propres généralisés stables de $A_S + \delta\mathrm{I}$.

On peut noter que d'après la Section 0.2.4.1, l'opérateur A_S engendre un semi–groupe analytique sur $\mathbb{H}_S$ et est à résolvante compacte. Ainsi, J_u est fini et pour tout $j \in \mathbb{N}$, $\mathrm{G}(\lambda_j)$ est de dimension finie. L'espace $\mathbb{Z}_u$ est donc de dimension finie que l'on note d_u.

Stabiliser le système (31) revient alors à stabiliser sa projection sur $\mathbb{Z}_u$:

$$\left\{ \begin{array}{l} \mathbf{z}_u'(t) = (A_u + \delta\mathrm{I})\mathbf{z}_u(t) + B_u \mathbf{h}(t), \quad t > 0, \\ \mathbf{z}_u(0) = \Pi_u \mathbf{z}_0, \end{array} \right. \tag{32}$$

où $\mathbf{z}_u = \Pi_u \mathbf{z}$, $A_u = \Pi_u A_S$, $B_u = \Pi_u B$ et où Π_u est la projection de $\mathbb{H}_S$ sur $\mathbb{Z}_u$ parallèle à $\mathbb{Z}_s$.

L'hypothèse $\boldsymbol{(\mathcal{H})}_\delta$ correspond à un test d'Hautus [150, Proposition 1.5.1] sur l'adjoint du système (32). Ainsi, sous cette hypothèse, le système (32) est contrôlable et donc stabilisable. On peut alors construire un opérateur de feedback qui le stabilise.

La méthode adoptée pour construire cet opérateur de feedback consiste à trouver $\mathcal{R}_\delta$ solution de l'équation de Riccati

$$\left\{ \begin{array}{l} \mathcal{R}_\delta = \mathcal{R}_\delta^* \geq 0, \\ \mathcal{R}_\delta(A_u + \delta\mathrm{I}) + (A_u^* + \delta\mathrm{I})\mathcal{R}_\delta + \mathrm{I} - \mathcal{R}_\delta B_u B_u^* \mathcal{R}_\delta = 0, \end{array} \right.$$

où l'opérateur de feedback $\mathcal{K}_\delta = -B_u^* \mathcal{R}_\delta \Pi_u$ permet de stabiliser le problème (32). On montre alors que l'opérateur en boucle fermée $A_S + \delta\mathrm{I} + B\mathcal{K}_\delta$ engendre un semi–groupe analytique stable sur $\mathbb{H}_S$. Ceci implique que le système en boucle fermée

$$\left\{ \begin{array}{l} \mathbf{z}'(t) = A_S \mathbf{z}(t) + B\mathcal{K}_\delta \mathbf{z}(t) + \mathbf{F}(t), \quad t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0, \end{array} \right. \tag{33}$$

admet une unique solution $\mathbf{z}$ dans $\{\mathbf{z} \text{ avec } e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0,\infty; D(A_S)) \cap \mathrm{H}^1(0,\infty;\mathbb{H}_S)\}$, pour tout $\mathbf{z}_0 \in \mathbb{V}_S$ et $\mathbf{F} \in \{\mathbf{f} \text{ avec } e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0,\infty;\mathbb{H}_S)\}$.

0.2.4.3 Étude du système linéarisé en boucle fermée

Dans cette section nous prouvons l'existence d'une unique solution au système linéaire en boucle fermée. Le système suivant correspond à (25) avec $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$

$$
\left\{
\begin{array}{ll}
\dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{w}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{w} - \mathbf{L_F}(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2, \mathbf{y}) - \nu\Delta\mathbf{v} + \nabla q = \mathbf{f} & \text{dans } (0,\infty)\times\mathscr{F}_s,\\
\operatorname{div} \mathbf{v} = 0 & \text{dans } (0,\infty)\times\mathscr{F}_s,\\
\mathbf{v} = \dot\theta_1 \partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y) + \dot\theta_2 \partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y) + \mathbf{g} & \text{sur } (0,\infty)\times\partial S_s,\\
\mathbf{v} = 0 & \text{sur } (0,\infty)\times\Gamma_{\mathrm{D}},\\
\sigma_F(\mathbf{v}, q)\mathbf{n} = 0 & \text{sur } (0,\infty)\times\Gamma_{\mathrm{N}},\\
\mathbf{v}(0,.) = \mathbf{v}_0(.) & \quad\text{dans } \mathscr{F}_s,\\
\mathcal{M}_{0,0}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = \begin{pmatrix}\displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v},q)\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)\\ \displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v},q)\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)\end{pmatrix} & \\
\qquad\qquad + \mathbf{L_S}(\theta_1,\theta_2) + \mathbf{s} + \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot\theta_1, \dot\theta_2) & \text{sur } (0,\infty),\\
\theta_1(0) = \theta_{1,0}, \qquad \theta_2(0) = \theta_{2,0}, & \\
\dot\theta_1(0) = \omega_{1,0}, \qquad \dot\theta_2(0) = \omega_{2,0}, &
\end{array}
\right.
\tag{34}
$$

où les termes $\mathbf{f}$, $\mathbf{g}$ et $\mathbf{s}$ appartiennent respectivement aux espaces

$$
\begin{aligned}
&\mathbb{F}^\infty_\delta = \{\mathbf{f} \;\text{ avec }\; e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathscr{F}_s))\},\\
&\mathbb{G}^\infty_\delta = \{\mathbf{g} \;\text{ avec }\; e^{\delta t}\mathbf{g} \in \mathrm{H}^1(0,\infty;\mathbf{H}^{3/2}(\partial S_s))\},\\
&\mathbb{S}^\infty_\delta = \{\mathbf{s} \;\text{ avec }\; e^{\delta t}\mathbf{s} \in \mathrm{L}^2(0,\infty;\mathbb{R}^2)\}.
\end{aligned}
$$

Comme en Section 0.1.5.3, nous recherchons une solution de (34) sous la forme $(\mathbf{v}, q, \theta_1, \theta_2) = (\overline{\mathbf{v}}, q, \theta_1, \theta_2) + (\mathbf{v_g}, 0, 0, 0)$, où $\mathbf{v_g}$ est un relèvement de $\mathbf{g}$

$$
\left\{
\begin{array}{ll}
\mathbf{v_g} = \mathbf{g} & \text{sur } (0,\infty)\times\partial S_s,\\
\operatorname{div} \mathbf{v_g} = 0 & \text{dans } (0,\infty)\times\mathscr{F}_s,\\
\mathbf{v_g} = 0 & \text{sur } (0,\infty)\times\Gamma_{\mathrm{D}},\\
(\nabla\mathbf{v_g} + \nabla\mathbf{v_g}^T)\mathbf{n} = 0 & \text{sur } (0,\infty)\times\Gamma_{\mathrm{N}}.
\end{array}
\right.
\tag{35}
$$

On montre alors que $\mathbf{z} = (\overline{\mathbf{v}}, \theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ est solution du problème (33) pour $\mathbf{z}_0$ et $\mathbf{F}$ bien choisis.

On peut ainsi, moyennant quelques détails techniques supplémentaires, construire une solution $(\mathbf{v}, q, \theta_1, \theta_2)$ au problème (34) qui appartient aux espaces suivants

$$
\begin{aligned}
&\mathbf{v} \in \mathbb{U}^\infty_\delta = \{\mathbf{v} \;\text{ avec }\; e^{\delta t}\mathbf{v} \in \mathrm{L}^2(0,\infty;\mathbf{H}^2_\beta(\mathscr{F}_s)) \cap \mathscr{C}^0([0,\infty);\mathbf{H}^1(\mathscr{F}_s)) \cap \mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathscr{F}_s))\},\\
&q \in \mathbb{P}^\infty_\delta = \{\mathbf{v} \;\text{ avec }\; e^{\delta t} q \in \mathrm{L}^2(0,\infty;\mathrm{H}^1_\beta(\mathscr{F}_s))\},\\
&(\theta_1, \theta_2) \in \Theta^\infty_\delta = \{(\theta_1,\theta_2) \;\text{ avec }\; e^{\delta t}(\theta_1,\theta_2) \in \mathrm{H}^2(0,\infty;\mathbb{R}^2)\},
\end{aligned}
$$

où $\mathbf{H}^2_\beta(\mathscr{F}_s)$ et $\mathrm{H}^1_\beta(\mathscr{F}_s)$ sont des espaces de Sobolev pondérés près des coins A et B (voir Figure 1), ils sont précisément définis en (2.26)–(2.27).

0.2.4.4 Traitement des termes non linéaires à l'aide d'un argument de point fixe

La dernière étape de la preuve du Théorème 0.2.2 consiste à traiter localement les termes non linéaires par un argument de point fixe. Soient l'espace

$$\mathbb{N}_\delta^\infty = \left\{ \begin{array}{l} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty \ \text{ avec } \ \forall t > 0, \ (\theta_1, \theta_2)(t) \in \mathbb{D}_\Theta \\ \qquad\qquad \theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, \quad \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0} \end{array} \right\},$$

et l'application $\Lambda^\infty : \mathbb{N}_\delta^\infty \to \mathbb{N}_\delta^\infty$ qui est définie par $\Lambda^\infty(\overline{\mathbf{v}}, \overline{q}, \overline{\theta_1}, \overline{\theta_2}) = (\mathbf{v}, q, \theta_1, \theta_2)$ où $(\mathbf{v}, q, \theta_1, \theta_2)$ est la solution du problème (34) avec

$$\begin{array}{l} \mathbf{f} = \mathbf{f}^{NL}(\overline{\theta_1}, \overline{\theta_2}, \overline{\mathbf{v}}, \overline{q}), \\ \mathbf{g} = \mathbf{g}^{NL}(\overline{\theta_1}, \overline{\theta_2}), \\ \mathbf{s} = \mathbf{s}^{NL}(\overline{\theta_1}, \overline{\theta_2}, \overline{\mathbf{v}}, \overline{q}), \end{array}$$

ces termes étant définis en (2.67).

Nous montrons, grâce à des estimations sur $\mathbf{f}^{NL}$, $\mathbf{g}^{NL}$, $\mathbf{s}^{NL}$, que pour R et ε suffisamment petits (ε est la taille maximale de la perturbation initiale), Λ^∞ est une contraction sur la boule

$$\mathbb{B}_R^\infty = \{(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{N}_\delta^\infty \ \text{ avec } \ \|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty} \leq R\}.$$

Le Théorème 0.2.2 est alors une conséquence du théorème du point fixe de Picard.

0.3 Simulations et stabilisation numériques du problème d'interaction fluide–structure

0.3.1 Présentation

On s'intéresse au système précédemment introduit en (22). On souhaite simuler numériquement la stabilisation par feedback de ce système et pouvoir observer le résultat établi dans le Théorème 0.2.2. Pour cela, il faut d'une part une méthode numérique capable de capturer l'évolution en temps du problème fluide–structure (15) et, d'autre part, il faut calculer une approximation de l'opérateur de feedback $\mathcal{K}_\delta$ et montrer que cette approximation est capable de stabiliser le système considéré. Ce dernier calcul est difficile car, sans traitement particulier, il nécessite l'inversion d'une matrice pleine de la taille du système global et la résolution d'une équation de Riccati de dimension le carré de la taille du système global. Notons que le contrôle que nous utilisons est calculé à partir du système linéarisé et que nous l'appliquons au système non linéaire. Dans ce chapitre, nous présentons les réponses que nous apportons à ces différentes difficultés ainsi que des résultats de simulations numériques originaux.

0.3.2 Résultats antérieurs

Nous présentons les travaux antérieurs concernant la simulation numérique d'un système d'interaction fluide–structure puis ceux concernant sa stabilisation numérique par feedback.

0.3.2.1 À propos de la simulation numérique d'un système d'interaction fluide–structure

La difficulté principale de la simulation numérique d'un système fluide–structure est que le domaine du fluide change au cours du temps. Pour des raisons de temps de calcul, nous préférons ne pas reconstruire entièrement le maillage à chaque pas de temps. Il faut donc utiliser des méthodes spécifiques qui permettent ou bien d'assurer que le maillage reste conforme au domaine sans générer trop de surcoût de calcul, ou alors utiliser une méthode qui fonctionne sur des maillages non conformes.

La première option est la plus présente dans la littérature, principalement sous le nom de méthode "Arbitrary Lagrangian Eulerian" (ALE) (voir par exemple [87, 100, 137, 139]). Elle permet avec un surcoût réduit de déformer le maillage pour coller au bord du domaine à chaque instant. Cependant, pour de grandes déformations du domaine fluide, elle peut conduire à un maillage de mauvaise qualité.

Une autre méthode consiste à mener les simulations dans le domaine fixe en résolvant numériquement (25)–(26) au lieu de (22). Cette méthode a pour inconvénient de contenir de très nombreux termes non linéaires du type de ceux de (26), elle a été néanmoins utilisée avec succès dans [116].

Enfin, la méthode que nous avons choisie repose sur l'utilisation de domaines fictifs. Ainsi, il n'est pas nécessaire que le maillage s'appuie sur la frontière du domaine et on peut donc choisir d'utiliser un maillage fixe (qui ne change pas au cours du temps). On peut citer une liste non exhaustive de travaux utilisant ce type de méthode [126, 55, 113, 75, 74, 82, 44, 40].

Étant donné que le maillage ne s'appuie pas sur la frontière du domaine, il faut modifier la formulation variationnelle du problème pour pouvoir imposer les conditions de Dirichlet sur le bord du domaine. Plus précisément, nous introduisons des multiplicateurs de Lagrange. Pour assurer la stabilité du schéma ainsi obtenu, nous devons ou bien choisir correctement les espaces d'approximation des différentes variables, ou bien ajouter au problème discrétisé un terme de stabilisation. Une alternative aux multiplicateurs de Lagrange est d'utiliser la méthode de Nitsche [119]. Celle–ci est cependant moins intéressante dans le cas d'un système d'interaction fluide–structure, en effet les multiplicateurs sont une variable utile du problème correspondant à la force exercée par le fluide sur la structure. Le paradigme consistant à combiner une méthode de domaines fictifs s'appuyant sur un maillage fixe avec des multiplicateurs de Lagrange, ou la méthode de Nitsche, a été utilisé pour résoudre des problèmes d'interaction fluide–structure dans [142, 3, 43, 109, 53]. Des avancées récentes dans ce domaine ont été présentées dans [27].

À notre connaissance, le contrôle par feedback d'un système fluide–structure simulé par des méthodes de type domaines fictifs est nouveau.

0.3.2.2 À propos du calcul numérique d'un contrôle par feedback

L'utilisation d'une équation de Riccati pour déterminer un contrôle est classique dans le cadre des équations différentielles ordinaires. Cependant, dans le cas d'une équation aux dérivées partielles, il est plus rare que cette méthode soit utilisée. En effet, la dimension de la solution du problème de Riccati correspond au carré de la dimension du problème semi–discrétisé et le système considéré est de grande taille, ce qui est prohibitif.

Pour pouvoir résoudre l'équation de Riccati, il faut une méthode de réduction de modèle, voir par exemple [2, 85, 116]. La matrice de feedback est calculée en suivant la même démarche que celle exposée au Chapitre 2 pour le système continu. On étudie la matrice associée aux équations linéarisées en domaine fixe (25) et on calcule la matrice de feedback à partir d'une équation de Riccati de petite dimension, grâce à une projection sur l'espace instable associé au problème. Ces travaux prouvent que la matrice ainsi calculée stabilise le système discrétisé. Dans [116], les simulations sont ensuite menées en domaine fixe.

Dans notre cas, nous utilisons cette méthode pour calculer la matrice de feedback et pour prouver qu'elle stabilise le système (25)–(26). Cependant, étant donné que nous voulons simuler le système en domaine mobile (22), nous ajoutons à la méthode précédente une manipulation qui permet à chaque instant de mener tous les calculs dans le domaine mobile $\mathscr{F}(\theta_1, \theta_2)$. Le seul calcul qui est effectué dans la domaine fixe $\mathscr{F}_s$ est celui de l'opérateur de feedback.

L'originalité de notre travail vient de la modélisation que nous utilisons pour la structure et du fait que l'on calcule l'évolution en temps du système fluide–structure en boucle fermée non pas dans un domaine fixe mais dans le domaine fluide au temps t. Par rapport à [116], nous proposons également une autre version de la démonstration du fait que la matrice de feedback stabilise le problème en domaine fixe.

0.3.3 Stabilisation du problème semi–discrétisé en domaine fixe

On s'intéresse à la discrétisation par éléments finis du problème (22). On considère une solution stationnaire de ce problème $(\mathbf{w}, p_{\mathbf{w}}, \xi_1, \xi_2) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s) \times \mathbb{D}_\Theta$, qui est solution du problème discrétisé correspondant à (23). Notons que contrairement au Chapitre 2, nous considérons que (ξ_1, ξ_2) est a priori non nul.

Stabiliser la solution de (22) autour de $(\mathbf{w}, p_{\mathbf{w}}, \xi_1, \xi_2)$ revient à stabiliser (25)–(26) autour de $(0,0,0,0)$. Dans la suite, nous nous intéressons au problème linéarisé, nous discrétisons donc le problème en domaine fixe (25). Les conditions de Dirichlet sont mises en œuvre en utilisant un multiplicateur de Lagrange $\boldsymbol{\lambda}$. On note $\mathbf{U}$ les coordonnées de l'approximation de $\mathbf{u}$, $\mathbf{P}$ celles de p et $\boldsymbol{\Lambda}$ celles de $\boldsymbol{\lambda}$. On regroupe l'état du système $\mathbf{z} = (\mathbf{U}^T \;\; \theta_1 \;\; \theta_2 \;\; \omega_1 \;\; \omega_2)^T$ et ses multiplicateurs $\boldsymbol{\eta} = (\mathbf{P}^T \;\; \boldsymbol{\Lambda}^T)^T$. Nous obtenons alors une approximation numérique de (22), avec des termes sources $\mathbf{f}$, $\mathbf{g}$ et $\mathbf{s}$ nuls, qui s'écrit sous la forme suivante

$$\left\{ \begin{array}{l} M_{\mathbf{zz}}\mathbf{z}'(t) = A_{\mathbf{zz}}\mathbf{z}(t) + A_{\mathbf{z}\boldsymbol{\eta}}\boldsymbol{\eta}(t) + B\mathbf{h}(t), \quad t \in [0,T], \\ A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}(t) = 0, \quad t \in [0,T], \\ \mathbf{z}(0) = \mathbf{z}_0, \end{array} \right. \tag{36}$$

où $M_{\mathbf{zz}} \in \mathbb{R}^{N_z \times N_z}$ est la matrice de masse du problème, $A_{\mathbf{zz}} \in \mathbb{R}^{N_z \times N_z}$ la matrice de rigidité, $A_{\mathbf{z}\boldsymbol{\eta}} \in \mathbb{R}^{N_z \times N_\eta}$ et $A_{\boldsymbol{\eta}\mathbf{z}} \in \mathbb{R}^{N_\eta \times N_z}$ sont des matrices couplant l'état $\mathbf{z} \in \mathbb{R}^{N_z}$ et le multiplicateur de Lagrange $\boldsymbol{\eta} \in \mathbb{R}^{N_\eta}$, enfin $B \in \mathbb{R}^{N_z \times 2}$ est la matrice de contrôle.

Dans la suite de ce chapitre, on se place dans un cadre où les équations discrétisées (36) sont bien posées, ce qui correspond à supposer une condition inf–sup portant sur les espaces des fonctions discrétisées. Cette condition sera explicitée dans le Chapitre 3. Elle implique que les matrices $A_{\mathbf{z}\boldsymbol{\eta}}$ et $A_{\boldsymbol{\eta}\mathbf{z}}$ sont de rang plein [60, Lemme A.40] (voir aussi [36]) et donc que la matrice $A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}}$ est inversible. Nous utilisons cette propriété par la suite.

0.3.3.1 Résultat

Le but de cette étude est de prouver que, dans le système (36), on peut choisir $\mathbf{h}$ sous la forme d'un feedback de manière à ce que la solution $\mathbf{z}$ décroisse vers 0 au cours du temps à vitesse exponentielle. Plus précisément, on choisit arbitrairement un taux de décroissance $\delta > 0$ et on suppose que l'hypothèse suivante est vérifiée.

Hypothèse $(\widetilde{\boldsymbol{\mathcal{H}}})_\delta$. Tout vecteur propre $(\mathbf{z}, \boldsymbol{\eta}) \in \mathbb{R}^{N_z} \times \mathbb{R}^{N_\eta}$ du problème adjoint de (36), associé à une valeur propre $\overline{\beta}$ telle que $\mathcal{R}e(\overline{\beta}) \geq -\delta$, qui appartient au noyau de l'adjoint de l'opérateur de contrôle est nul. C'est–à–dire

$$\begin{pmatrix} A_{\mathbf{zz}}^T & A_{\boldsymbol{\eta}\mathbf{z}}^T \\ A_{\mathbf{z}\boldsymbol{\eta}}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{pmatrix} = \overline{\beta} \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{pmatrix} \quad \text{and} \quad B^T\mathbf{z} = 0 \quad \Longrightarrow \quad (\mathbf{z}, \boldsymbol{\eta}) = 0.$$

Cette hypothèse, vérifiable numériquement, correspond à la version discrétisée de l'hypothèse $(\boldsymbol{\mathcal{H}})_\delta$. En l'admettant, on peut montrer le résultat suivant.

Théorème 0.3.1 (Théorème 3.3.19 du Chapitre 3)**.** *Pour $N_z > 0$ fixé. Soit $\delta > 0$ tel que l'hypothèse $(\widetilde{\mathcal{H}})_\delta$ soit vérifiée. Alors on peut construire une matrice $\mathbf{K}_{\delta,\omega} \in \mathbb{R}^{2\times N_z}$ telle que pour toutes données $\mathbf{z}_0$ vérifiant les conditions de compatibilité $A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}_0 = 0$, une solution du système* (36) *avec le contrôle donné sous la forme d'un feedback*

$$\mathbf{h}(t) = \mathbf{K}_{\delta,\omega}\mathbf{z}(t),$$

décroît exponentiellement au cours du temps avec un taux δ arbitraire :

$$\forall t \in (0, T], \quad |\mathbf{z}(t)| \leq Ce^{-\delta t}.$$

Ce résultat est l'analogue pour le problème discrétisé de la Proposition 2.2.1. Notons que la condition de compatibilité $A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}_0 = 0$ correspond à une version discrétisée de la condition de compatibilité continue en domaine fixe.

0.3.3.2 Plan de la preuve du Théorème 0.3.1

La preuve du Théorème 0.3.1 comporte les étapes suivantes :
— on projette d'abord (36) selon une projection Π définie en (37) afin d'éliminer les multiplicateurs $\boldsymbol{\eta}$,
— on effectue ensuite la décomposition spectrale des matrices obtenues,
— enfin, la matrice de feedback est obtenue à partir de la solution d'une équation de Riccati de petite dimension.

Notons que la preuve suit les mêmes étapes que celle du Théorème 0.2.2, à l'exception de l'argument de point fixe que nous n'utilisons pas ici puisque notre résultat porte sur un système linéaire. Étendre le résultat au cas non linéaire ne devrait pas poser de difficulté.

Étape 1 : Problème projeté direct.
Nous voulons écrire un système équivalent à (36) qui ne porte que sur l'état $\mathbf{z}$, c'est–à–dire qui ne fait pas intervenir les multiplicateurs $\boldsymbol{\eta}$. Pour cela, nous définissons la projection Π de $\mathbb{R}^{N_z}$ sur $\mathrm{Ker}(A_{\boldsymbol{\eta}\mathbf{z}})$ parallèlement à $\mathrm{Im}(M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})$,

$$\Pi = \mathrm{I} - M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}. \tag{37}$$

Il s'agit d'une approximation numérique de $\Pi_{\mathbb{H}_S}$ (la projection orthogonale sur $\mathbb{H}_S$ défini en (30)) et nous étudions le système projeté

$$\begin{cases} \Pi\mathbf{z}'(t) = \mathbb{A}\Pi\mathbf{z}(t) + \mathbb{B}\mathbf{h}(t), \quad t \in [0, T], \\ (\mathrm{I} - \Pi)\mathbf{z}(t) = 0, \quad t \in [0, T], \\ \Pi\mathbf{z}(0) = \Pi\mathbf{z}_0, \end{cases} \tag{38}$$

où $\mathbb{A} = \Pi M_{\mathbf{zz}}^{-1}A_{\mathbf{zz}}$ et $\mathbb{B} = \Pi M_{\mathbf{zz}}^{-1}B$. Pour $\mathbf{z}_0$ vérifiant la condition de compatibilité $A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}_0 = 0$, le problème (38) est équivalent au problème (36) au sens où pour toute solution $\Pi\mathbf{z}$ de (38), il existe $\boldsymbol{\eta}$ choisi convenablement tel que $(\Pi\mathbf{z}, \boldsymbol{\eta})$ est solution de (36). La réciproque est vraie.

Pour pouvoir appliquer les résultats classiques de la littérature en contrôle, par exemple [24, 54], nous devons travailler avec le système projeté (38) car la contrainte $A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z} = 0$ ainsi que le multiplicateur de Lagrange $\boldsymbol{\eta}$ n'y apparaissent pas. C'est donc ce que nous faisons dans la suite de la preuve.

Étape 1 bis : Problème projeté adjoint.
On peut de la même façon montrer que le problème adjoint

$$\begin{cases} -M_{\mathbf{zz}}\mathbf{\Phi}'(t) = A_{\mathbf{zz}}^T\mathbf{\Phi}(t) + A_{\boldsymbol{\eta}\mathbf{z}}^T\boldsymbol{\zeta}(t), \quad t \in [0,T], \\ A_{\mathbf{z}\boldsymbol{\eta}}^T\mathbf{\Phi}(t) = 0, \quad t \in [0,T], \\ \mathbf{\Phi}(T) = \mathbf{\Phi}_T, \end{cases} \tag{39}$$

est équivalent au système projeté

$$\begin{cases} -\widetilde{\Pi}\mathbf{\Phi}'(t) = \mathbb{A}^{\#}\widetilde{\Pi}\mathbf{\Phi}(t), \quad t \in [0,T], \\ (\mathrm{I} - \widetilde{\Pi})\mathbf{\Phi}(t) = 0, \quad t \in [0,T], \\ \widetilde{\Pi}\mathbf{\Phi}(T) = \widetilde{\Pi}\mathbf{\Phi}_T, \end{cases} \tag{40}$$

où $\widetilde{\Pi}$ est un projecteur associé au problème adjoint et $\mathbb{A}^{\#} = \widetilde{\Pi}M_{\mathbf{zz}}^{-1}A_{\mathbf{zz}}^T$.

Étape 2 : Décomposition spectrale des opérateurs.
On considère les vecteurs propres de $\mathbb{A}$, c'est–à–dire les vecteurs $\mathbf{f}$ de $\mathbb{C}^{N_z}$ vérifiant

$$\begin{cases} \mathbb{A}\mathbf{f} = \beta\mathbf{f}, \\ A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{f} = 0, \end{cases} \tag{41}$$

où β est la valeur propre associée à ce vecteur. On considère également les vecteurs propres généralisés de $\mathbb{A}$ associés à une solution $(\mathbf{f}, \beta)$ de (41) et à un entier naturel k, c'est–à–dire les vecteurs $\mathbf{f}^k$ de $\mathbb{C}^{N_z}$ vérifiant

$$\begin{cases} (\mathbb{A} - \beta\mathrm{I})^k\mathbf{f}^k = \mathbf{f}, \\ A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{f}^k = 0. \end{cases} \tag{42}$$

On construit une matrice Ψ dont les colonnes sont les vecteurs propres et vecteurs propres généralisés de $\mathbb{A}$. On construit aussi de la même façon une matrice $\widetilde{\Psi}$ dont les colonnes sont exactement les vecteurs propres et vecteurs propres généralisés de $\mathbb{A}^{\#}$.

Les colonnes de Ψ (respectivement $\widetilde{\Psi}$) forment une base de $\mathrm{Im}(\Pi)$ (respectivement $\mathrm{Im}(\widetilde{\Pi})$). De plus, quitte à les réordonner et à les normaliser, on montre les relations

$$\Lambda_{\mathbb{C}} = \widetilde{\Psi}^T M_{\mathbf{zz}}\mathbb{A}\Psi \qquad \text{et} \qquad \Lambda_{\mathbb{C}}^T = \Psi^T M_{\mathbf{zz}}\mathbb{A}^{\#}\widetilde{\Psi},$$

où $\Lambda_{\mathbb{C}}$ est une matrice diagonale par blocs issue de la décomposition de Jordan de $\mathbb{A}$, et on a la relation de biorthogonalité

$$\Psi^T M_{\mathbf{zz}}\widetilde{\Psi} = \mathrm{I}_{N_\pi},$$

où N_π est la dimension de $\mathrm{Im}(\Pi)$.

Dans la suite, on parlera de vecteurs propres généralisés pour désigner aussi bien les solutions de (41) que celles de (42). Les vecteurs propres généralisés de $\mathbb{A}$ et $\mathbb{A}^{\#}$ sont a priori à valeurs complexes et les matrices Ψ et $\widetilde{\Psi}$ le sont aussi. Nous introduisons les matrices E et $\widetilde{E}$ qui sont à valeur réelle et dont les colonnes sont obtenues à partir des parties réelle et imaginaire des colonnes de Ψ et $\widetilde{\Psi}$. Ces matrices vérifient les propriétés suivantes

- E est une base de $\mathrm{Im}(\Pi)$,
- $\widetilde{E}$ est une base de $\mathrm{Im}(\widetilde{\Pi})$,
- on a les décompositions

$$\Lambda_{\mathbb{R}} = \widetilde{E}^T M_{\mathbf{zz}}\mathbb{A}E \qquad \text{et} \qquad \Lambda_{\mathbb{R}}^T = E^T M_{\mathbf{zz}}\mathbb{A}^{\#}\widetilde{E},$$

  où $\Lambda_{\mathbb{R}}$ est une matrice réelle diagonale par blocs,

— on a la relation de biorthogonalité

$$E^T M_{\mathbf{zz}} \widetilde{E} = \mathrm{I}_{N_\pi}.$$

Les matrices E et $\widetilde{E}$ sont utilisées dans la suite de la preuve. Le lecteur pourra trouver plus d'informations dans la Section 3.3.3.

Étape 3 : Projection du problème sur un espace de petite dimension. On note (β_j) les valeurs propres de $\mathbb{A}$ et J_u l'ensemble d'indices j associés à des valeurs propres β_j vérifiant $\mathcal{R}e(\beta_j) \geq -\delta$. On note $\mathrm{G}(\beta_j)$ l'espace propre généralisé de $\mathbb{A}$ associé à la valeur propre β_j. On construit alors l'espace engendré par les vecteurs propres et vecteurs propres généralisés instables de $\mathbb{A} + \delta \mathrm{I}$:

$$\mathbb{Z}_u = \bigoplus_{j \in J_u} \mathrm{G}(\beta_j).$$

On note d_u la dimension de $\mathbb{Z}_u$. De la même façon, on peut définir $\mathbb{Z}_s$ l'espace engendré par les vecteurs propres généralisés stables de $\mathbb{A} + \delta \mathrm{I}$,

$$\mathbb{Z}_s = \bigoplus_{j \in N_z \setminus J_u} \mathrm{G}(\beta_j),$$

on a alors la décomposition

$$\mathrm{Im}(\Pi) = \mathbb{Z}_u \bigoplus \mathbb{Z}_s.$$

Stabiliser (38) revient à stabiliser sa projection sur l'espace instable $\mathbb{Z}_u$:

$$\left\{ \begin{array}{l} \mathbf{z}_u'(t) = A_u \mathbf{z}_u(t) + B_u \mathbf{h}(t), \quad t \in [0, T], \\ \mathbf{z}_u(0) = \Pi_u \mathbf{z}_0, \end{array} \right. \tag{43}$$

où $\mathbf{z}_u = \Pi_u \mathbf{z}$, $A_u = \Pi_u \mathbb{A}$, $B_u = \Pi_u \mathbb{B}$ et où Π_u est la projection de $\mathbb{R}^{N_z}$ sur $\mathbb{Z}_u$ parallèlement à $\mathbb{Z}_s \bigoplus \mathrm{Ker}(\Pi)$. On peut de plus écrire Π_u à partir des familles introduites précédemment

$$\Pi_u = E_u \widetilde{E}_u^T M_{\mathbf{zz}},$$

où les familles E et $\widetilde{E}$ ont été décomposées en $E = [E_u \ E_s]$ et $\widetilde{E} = [\widetilde{E}_u \ \widetilde{E}_s]$, E_u contenant les vecteurs de E dans $\mathbb{Z}_u$ et E_s ses vecteurs dans $\mathbb{Z}_s$, et de même pour $\widetilde{E}$. Notons que les colonnes de E sont soit dans $\mathbb{Z}_u$, soit dans $\mathbb{Z}_s$. On peut montrer une propriété similaire pour $\widetilde{E}$.

Étape 4 : Construction d'une matrice de feedback stabilisante. L'hypothèse $(\widetilde{\boldsymbol{\mathcal{H}}})_\delta$ correspond à un test d'Hautus pour le système adjoint de (43), elle implique donc que (43) est stabilisable. On peut donc construire une matrice de feedback stabilisant le problème (43) comme

$$\mathbf{K}_{\delta,\omega} = -B_u^T \mathcal{R}_{\delta,\omega} \Pi_u, \tag{44}$$

où on trouve $\mathcal{R}_{\delta,\omega} \in \mathbb{R}^{N_z \times N_z}$ en résolvant l'équation de Riccati

$$\left\{ \begin{array}{l} \mathcal{R}_{\delta,\omega} = \mathcal{R}_{\delta,\omega}^T \geq 0, \\ \mathcal{R}_{\delta,\omega}(A_u + \omega \mathrm{I}) + (A_u^T + \omega \mathrm{I}) \mathcal{R}_{\delta,\omega} + \Pi_u^T \Pi_u - \mathcal{R}_{\delta,\omega} B_u B_u^T \mathcal{R}_{\delta,\omega} = 0. \end{array} \right. \tag{45}$$

Dans cette équation $\omega > \delta$ est un paramètre permettant de régler la force du contrôle. Plus ω est grand, plus le contrôle par feedback sera fort.

Nous pouvons ensuite prouver que le contrôle $\mathbf{h} = \mathbf{K}_{\delta,\omega} \mathbf{z}$ calculé à l'aide du feedback $\mathbf{K}_{\delta,\omega}$ stabilise le problème (38) et donc aussi le problème (36). Ceci prouve le Théorème 0.3.1.

0.3.4 Calcul numérique de la matrice de feedback

Nous traitons dans cette section des difficultés liées à la mise en œuvre numérique du calcul de la matrice de feedback $\mathbf{K}_{\delta,\omega}$. L'expression proposée pour $\mathbf{K}_{\delta,\omega}$ en (44) nécessite le calcul de Π, ce qui demande l'inversion de la matrice $A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}$ (voir (37)), qui est une matrice pleine de taille $\mathbb{R}^{N_z\times N_z}$. Cette opération nécessite un temps de calcul très important et il est donc préférable de trouver une autre stratégie pour résoudre les problèmes aux valeurs propres généralisés dont nous avons besoin pour calculer les matrices E et $\widetilde{E}$. Ce point est encore plus critique en dimension 3 que nous ne traitons pas dans cette étude.

De plus, la résolution de l'équation de Riccati (45), dont la solution $\mathcal{R}_{\delta,\omega}$ appartient à $\mathbb{R}^{N_z\times N_z}$, a un coût prohibitif. Dans cette section, nous présentons une technique basée sur une réduction de modèle permettant de calculer $\mathbf{K}_{\delta,\omega}$ à moindre coût.

0.3.4.1 Réécriture des problèmes aux valeurs propres

Nous pouvons calculer les valeurs propres β_j de $\mathbb{A}$, ainsi que ses vecteurs propres et vecteurs propres généralisés, de manière plus efficace en considérant le problème aux valeurs propres

$$\begin{pmatrix} A_{\mathbf{zz}} & A_{\mathbf{z}\boldsymbol{\eta}} \\ A_{\boldsymbol{\eta}\mathbf{z}} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{f} \\ \boldsymbol{\eta}_{\mathbf{f}} \end{pmatrix} = \beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{f} \\ \boldsymbol{\eta} \end{pmatrix}, \tag{46}$$

au lieu de (41).

Le point fondamental de cette section est que les deux problèmes (41) et (46) sont mathématiquement équivalents. Cependant, le calcul de $\mathbb{A}$ nécessite le calcul de Π et donc l'inversion de la matrice $A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}$, qui est pleine. Ainsi, nous préférons résoudre le problème aux valeurs propres (46), la résolution de (41) n'étant pas envisageable.

Une équivalence similaire est établie pour le problème aux valeurs propres de $\mathbb{A}^{\#}$ et pour les problèmes aux valeurs propres généralisés.

0.3.4.2 Réécriture de la matrice de feedback $\mathbf{K}_{\delta,\omega}$

L'expression (44) ne permet pas un calcul efficace de $\mathbf{K}_{\delta,\omega}$. En effet, la résolution de l'équation de Riccati (45), dont la solution appartient à $\mathbb{R}^{N_z\times N_z}$, est bien trop coûteuse en pratique. Il est donc préférable de calculer $\mathbf{K}_{\delta,\omega}$ comme suit

$$\mathbf{K}_{\delta,\omega} = -B^T\widetilde{E}_u\widetilde{\mathcal{R}}_{\delta,\omega}\widetilde{E}_u^T M_{\mathbf{zz}}, \tag{47}$$

où $\widetilde{\mathcal{R}}_{\delta,\omega} \in \mathbb{R}^{d_u\times d_u}$ est la solution de l'équation

$$\left\{\begin{array}{l} \widetilde{\mathcal{R}}_{\delta,\omega} = \widetilde{\mathcal{R}}_{\delta,\omega}^T \geq 0, \\ \widetilde{\mathcal{R}}_{\delta,\omega}(\Lambda_u + \omega \mathrm{I}_{\mathbb{R}^{d_u}}) + (\Lambda_s + \omega \mathrm{I}_{\mathbb{R}^{d_u}})\widetilde{\mathcal{R}}_{\delta,\omega} + E_u^T E_u + \widetilde{\mathcal{R}}_{\delta,\omega}\widetilde{E}_u^T BB^T\widetilde{E}_u\widetilde{\mathcal{R}}_{\delta,\omega} = 0. \end{array}\right. \tag{48}$$

Nous insistons sur le fait que la solution $\widetilde{\mathcal{R}}_{\delta,\omega}$ de l'équation (48) appartient à $\mathbb{R}^{d_u\times d_u}$, elle est donc de dimension bien plus petite que $\mathcal{R}_{\delta,\omega}$ solution de (45) qui appartient à $\mathbb{R}^{N_z\times N_z}$ (en pratique, $d_u < 10$). La petite dimension de $\widetilde{\mathcal{R}}_{\delta,\omega}$ est un point essentiel de la méthode de calcul puisqu'elle permet la résolution de l'équation (48), qui est alors très simple, en un temps très raisonnable.

En manipulant les équations (45) et (48), on peut montrer que $\mathcal{R}_{\delta,\omega}$ est solution de (45) si et seulement si $\widetilde{\mathcal{R}}_{\delta,\omega} = E_u^T\mathcal{R}_{\delta,\omega}E_u$ est solution de (48). Ainsi, les deux expressions (44) et (47) proposées pour $\mathbf{K}_{\delta,\omega}$ coïncident. Pour plus de détails, on peut se reporter à la Section 3.4.

0.3.5 Simulations numériques du système fluide–structure

La méthode utilisée repose sur des domaines fictifs, l'ensemble du domaine Ω, incluant la partie fluide et la structure, est maillé. L'interface $\partial S(\theta_1, \theta_2)$ entre le fluide et la structure peut a priori couper les éléments du maillage. On illustre ci–dessous une telle méthode (Figure 5) pour un maillage cartésien. En pratique, nous utiliserons un maillage triangulaire.

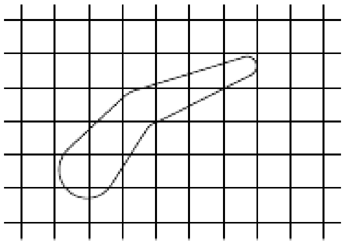


FIGURE 5 – Le domaine fictif.

On s'appuie sur la méthode présentée pour un cadre stationnaire dans [52, 65] ou instationnaire dans [53]. Les conditions de Dirichlet sont prises en compte par des multiplicateurs de Lagrange. Ces mêmes multiplicateurs de Lagrange sont ensuite utilisés pour calculer la force exercée par le fluide sur la structure. Pour obtenir une valeur plus précise du multiplicateur de Lagrange, on ajoute à la formulation variationnelle associée au problème (22) un terme de stabilisation sur le bord de la structure et on fait une correction sur les éléments dont la partie appartenant au domaine fluide est trop petite (voir Chapitre 3, Section 3.6.2.3). Le terme de stabilisation permet également d'améliorer le conditionnement du problème.

Le contrôle $\mathbf{h}$ est calculé à partir de l'état du système et de la matrice de feedback que l'on a précédemment construite :

$$\mathbf{h}(t) = \mathbf{K}_{\delta,\omega}(\mathbf{v}, \theta_1 - \xi_1, \theta_2 - \xi_2, \dot{\theta}_1, \dot{\theta}_2),$$

où $\mathbf{v}(t, \mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}^\mathbf{S}}(\theta_1(t), \theta_2(t), \mathbf{y}))^T \mathbf{u}(t, \mathbf{\Phi}^\mathbf{S}(\theta_1(t), \theta_2(t), \mathbf{y})) - \mathbf{w}(\mathbf{y})$.

Tous les calculs sont effectués avec la bibliothèque libre GetFEM++ écrite en C++. Un des avantages de cette approche est de pouvoit utiliser tous les outils qui y sont développés. En particulier, nous pourrons utiliser la calcul parallèle avec MPI si nous souhaitons poursuivre cette étude en 3D.

Le calcul de la marche en temps se fait de manière partitionnée. Nous mettons d'abord à jour la position de la structure, puis nous calculons le nouvel état du fluide. À chaque pas de temps,

— nous calculons la valeur du contrôle $\mathbf{h}$ (lorsque nous sommes en train de simuler l'évolution du système en boucle fermée),
— nous calculons la valeur des paramètres de la structure,
— nous mettons à jour la position de l'interface fluide–structure, ainsi que les méthodes d'intégration qui en dépendent,
— nous calculons l'état du fluide.

Bien que ce travail de thèse ne contienne pas de réelle nouveauté concernant le fonctionnement de la méthode numérique employée, la mise en œuvre faite dans le cadre de la modélisation considérée a demandé un réel travail de développement d'outils numériques. Parmi ces outils, nous avons développé une méthode de calcul numérique du difféomorphisme $\mathbf{\Phi}^\mathbf{S}(\theta_1, \theta_2, .)$ et de son inverse $\mathbf{\Phi}^\mathbf{S}(\theta_1, \theta_2.)^{-1}$ qui, dans le Chapitre 2, ne sont pas donnés par une expression explicite. Nous avons de plus développé une méthode permettant de calculer précisément la valeur de la fonction level–set en s'appuyant sur une liste de points positionnés exactement sur l'interface entre le fluide et la structure.

Les efforts consentis pour développer ces outils leur permettent d'être relativement génériques et donc d'être réutilisables pour des géométries différentes.

De plus, de nombreuses améliorations du code permettent de réduire les coûts de calcul. Par exemple, à chaque pas de temps, les coefficients des matrices correspondant à des régions loin de la structure, et qui donc ne changent pas, n'ont pas à être réassemblés.

Notons que le feedback calculé numériquement est a priori utilisable pour stabiliser le système continu tant que le maillage utilisé pour la discrétisation est assez fin.

0.4 Perspectives

Plusieurs perspectives peuvent être envisagées suite à ce travail de thèse.

— Tout d'abord, il serait intéressant d'essayer de traiter le cas où la structure est constituée d'un nombre fini de solides rigides reliés entre eux par des liaisons mécaniques (voir Figure 2a pour le cas d'une liaison pivot). Ce type de structure est courant dans le monde industriel. Comme nous l'avons évoqué dans la section de modélisation, dans ce cas, la vitesse peut être discontinue à l'interface entre les différents solides. Du point de vue de la modélisation, cela revient à retirer l'hypothèse (6). Cette hypothèse est essentielle dans l'approche que nous avons utilisée. En effet, combinée à la condition d'adhérence (8), elle impose à la vitesse du fluide d'être très régulière sur $\partial S(\theta_1, \theta_2)$.

Si cette donnée est discontinue, alors nous ne pouvons pas utiliser la même approche. En effet, si la trace de la vitesse sur $\partial S(\theta_1, \theta_2)$ est discontinue, alors la vitesse à l'instant t ne peut pas appartenir à l'espace de Sobolev $\mathbf{H}^2(\mathscr{F}(\theta_1, \theta_2))$. Il faudrait donc changer le cadre fonctionnel que nous utilisons pour pouvoir traiter ce cas. On pourrait par exemple envisager de travailler avec des solutions faibles comme dans [11, 114, 112]. Une autre piste serait de régulariser l'assemblage de solides rigides en une structure se déformant continûment avec un paramètre de régularisation $\varepsilon > 0$ et de retrouver l'assemblage de solides quand $\varepsilon \to 0$. On ménerait alors l'analyse sur le problème ayant un champ de vitesse continu et on pourrait essayer de retrouver des estimées par passage à la limite sur l'assemblage de solides.

— Une autre suite à ce travail serait de travailler sur l'hypothèse $(\boldsymbol{\mathcal{H}})_\delta$. Nous avons déjà montré qu'elle était intrinsèque au modèle considéré. Cette hypothèse est une condition essentielle à la preuve de la stabilisation du système. Il serait donc bon d'essayer de déterminer dans quelles conditions cette hypothèse est vérifiée. On peut par exemple se demander si cette hypothèse est satisfaisante génériquement par rapport au domaine.

De la même façon, on pourrait étudier l'hypothèse $\widetilde{(\boldsymbol{\mathcal{H}})}_\delta$. En particulier, on pourrait se demander pour quels espaces d'approximation l'hypothèse continue $(\boldsymbol{\mathcal{H}})_\delta$ implique l'hypothèse discrétisée $\widetilde{(\boldsymbol{\mathcal{H}})}_\delta$.

— Un autre axe d'approfondissement serait de stabiliser le système par feedback avec une information partielle uniquement. On pourrait, pour calculer le contrôle, au lieu de se donner l'état complet du système, ne considérer qu'une mesure partielle, par exemple l'état de la structure. Il faudrait alors construire un observateur pour estimer l'état global du système. Le contrôle serait alors calculé à partir de l'état estimé en utilisant l'opérateur de feedback $\mathcal{K}_\delta$ que nous avons construit dans le Chapitre 2. Notons que cette étude pourrait être menée aussi bien sur le système continu du Chapitre 2 que sur le système discrétisé du Chapitre 3. Cette question a déjà été abordée pour d'autres modèles. On citera notamment les constructions d'observateurs fournies par [49] pour un modèle de réaction–diffusion, par [45, 78, 86, 90, 115] pour un fluide seul et [25, 26] pour le cas d'un système d'interaction fluide–structure.

— On pourrait étudier la convergence de la matrice de feedback $\mathbf{K}_{\delta,\omega}$ du système discrétisé vers l'opérateur de feedback $\mathcal{K}_\delta$ du système continu quand le pas de maillage tend vers zéro. Cela permettrait d'estimer le raffinement de maillage nécessaire pour approcher correctement $\mathcal{K}_\delta$ par $\mathbf{K}_{\delta,\omega}$ et ainsi utiliser ce dernier pour stabiliser le système continu.

— Ensuite, mener la même étude en considérant des angles sur la géométrie de la structure viendrait renforcer la pertinence des résultats que nous avons présentés. En effet, les profils d'aile d'avion classiques possèdent de tels angles à l'arrière (leur bord de fuite), par exemple un profil NACA est représenté sur la Figure 6.

FIGURE 6 – Un profil d'aile de type NACA.

Traiter la présence d'un angle sur la structure représente une réelle difficulté aussi bien du point de vue théorique que numérique. En effet, d'un point de vue théorique, un angle sur la structure est en mouvement, il est donc plus difficile à traiter que les angles aux coins du domaine Ω. Par ailleurs, d'un point de vue numérique, il est bon que le maillage s'appuie sur l'angle au bout du profil. Il faudrait donc adapter le code pour permettre cela.

— Nous pourrions également étendre les outils numériques que nous avons développés. Le cas d'une structure dépendant de plus de deux paramètres serait une première extension. Elle permettrait de traiter des structures déformables disposant d'une plus grande liberté de déformation. Dans un second temps, étudier le problème en dimension trois comporterait des difficultés liées au fait qu'il faudrait résoudre des systèmes de grande taille. Ainsi, de nouvelles problématiques liées au calcul haute performance devraient être abordées. L'étude théorique que nous avons menée, ainsi que les outils numériques que nous avons développés pourraient être réutilisés pour traiter ces problèmes (voir les hypothèses faites sur $\mathbf{X}$ lors de la modélisation et les hypothèses faites sur les espaces d'approximation dans le Chapitre 3).

— Enfin, on pourrait mener une analyse numérique propre de l'algorithme de simulation utilisé dans le Chapitre 3. Ceci permettrait de garantir la convergence de la solution approchée vers la solution continue. Cette analyse présente plusieurs difficultés. D'abord, il faut être capable de traiter le fait que le couplage fluide–structure est mis en œuvre de manière partitionnée. On peut se reporter pour cela à [62]. De plus, la présence d'éléments coupés complique l'analyse. On pourra s'inspirer des études [52, 65] analysant la convergence de la solution pour le problème de Stokes stationnaire, et de [121, 120] où l'analyse d'un problème instationnaire différent du nôtre a déjà été menée pour des méthodes non conformes.

Chapitre 1

Existence de solutions fortes au problème d'interaction fluide–structure en temps petits

Abstract. We study the existence of strong solutions to a 2d fluid–structure system. The fluid is modelled by the incompressible Navier–Stokes equations. The structure represents a steering gear and is described by a finite number of parameters and its equations are derived from a virtual work principle. The global domain represents a wind tunnel and imposes mixed boundary conditions to the fluid velocity. Our method reposes on the analysis of the linearized system. Under compatibility conditions on the initial conditions, we can guarantee local existence in time of strong solutions to the fluid–structure problem.

MSC numbers : 74F10, 74H20, 74H25, 74H30, 76D03.

1.1 Introduction

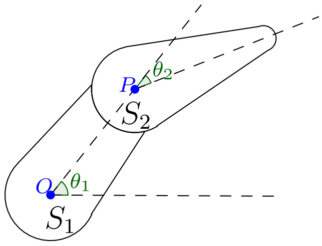
The goal of this study is to prove the existence of a solution to a fluid–structure problem. The fluid is modelled by the incompressible Navier–Stokes equations and the structure, immersed in the fluid, is governed by a finite number of parameters.

For the sake of simplicity, only two parameters θ_1 and θ_2 are considered to describe the motion of the structure. However, all results remain valid for any finite number of parameters.

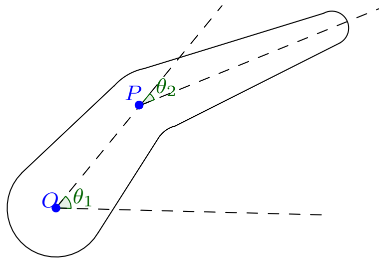
1.1.1 Modelling of the problem

The considered structure lies inside an open bounded domain $\Omega \subset \mathbb{R}^2$ and deforms itself over time. The couple of parameters (θ_1, θ_2) lies in an admissible domain $\mathbb{D}_\Theta$ which is an open connected subset of $\mathbb{R}^2$. Let S_{ref}, a smooth closed connected subset of Ω, be the reference configuration for the structure (for instance S_{ref} is the volume occupied by the structure for $\theta_1 = \theta_2 = 0$). We consider a function $\mathbf{X}$ defined on $\mathbb{D}_\Theta \times S_{\text{ref}}$ that computes the position in the structure according to the reference position in S_{ref} and to the value of the parameters $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$.

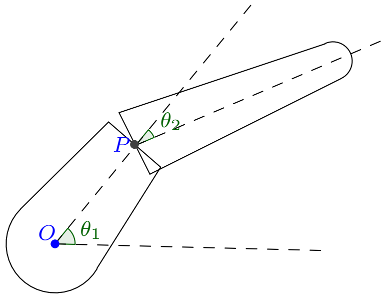
The volume occupied by the structure for the parameters $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$ is a closed bounded connected subset of Ω denoted $S(\theta_1, \theta_2) = \mathbf{X}(\theta_1, \theta_2, S_{\text{ref}})$. We further assume that for every



(a) A pivoting link P.



(b) A structure continuously deforming.



(c) A limit configuration.

FIGURE 1.1 – Three different kinds of structure deformation.

$(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, $S(\theta_1, \theta_2) \subset \Omega$, i.e. there is no contact between the structure $S(\theta_1, \theta_2)$ and the boundary of the domain $\partial\Omega$.

1.1.1.1 Motivations

Structures depending only on a finite number of parameters arise in the field of aeronautics. For instance, let us consider a steering gear structure. In a first approach, we can model this structure by two rigid solids. Solid S_1 is tied to the fixed frame by a pivoting link O and solid S_2 is tied to solid S_1 by a pivoting link P. The whole model is represented in Fig. 1.1a. Note that S_1 can be thought of as the aerofoil of a wing and S_2 as a steering gear such as an aileron. For a given S_{ref}, the function $\mathbf{X}^a$ representing the motion of this structure with respect to (θ_1, θ_2) is given below

$$\mathbf{X}^a(\theta_1, \theta_2, \mathbf{y}) = \chi_{S_1}(\mathbf{y}) R_{\theta_1}\mathbf{y} + \chi_{S_2}(\mathbf{y})(R_{\theta_1}\mathbf{y}_P^{\text{ref}} + R_{\theta_1+\theta_2}(\mathbf{y} - \mathbf{y}_P^{\text{ref}})), \quad \forall \mathbf{y} \in S_{\text{ref}}, \quad \forall (\theta_1, \theta_2) \in \mathbb{D}_\Theta,$$

where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation matrix of angle θ, $\mathbf{y}_P^{\text{ref}} = (y_{P,1}, y_{P,2})^T$ is the coordinate of the point P in the reference configuration S_{ref} and χ_E is the characteristic function over a set $E \subset \mathbb{R}^2$ given below

$$\forall \mathbf{y} \in \Omega, \quad \chi_E(\mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{y} \in E, \\ 0 & \text{else.} \end{cases} \tag{1.1}$$

In the previous example, the domain of definition $\mathbb{D}_\Theta$ of (θ_1, θ_2) is chosen such that no overlaps of the structure occur.

Note that for $\theta_2 \neq 0$, the function $\mathbf{X}^a(\theta_1, \theta_2, .)$ is not a diffeomorphism as it is discontinuous through the interface $\partial S_1 \cap \partial S_2$ between the two solids. In the same way, for $\dot{\theta}_2 \neq 0$, the velocity field is discontinuous inside the structure (we denote $\dot{\theta}_2$ the time derivative of θ_2). In other words, if we keep S_1 at rest and rotate S_2 around P, a discontinuity of the velocity appears through the interface between the two solids. This discontinuity can reduce the regularity expected for the fluid velocity. Indeed, if we assume no–slip boundary conditions between the fluid and the structure and if at time t the trace of the velocity is discontinuous on $\partial S(\theta_1(t), \theta_2(t))$, then a Sobolev injection argument shows that we cannot hope for a better regularity in space for the velocity of the fluid than the Sobolev space $\mathrm{L}^2(0,T;\mathbf{H}^1(\Omega\backslash S(\theta_1(t),\theta_2(t))))$ [1], while for strong solutions we usually expect the velocity in the Sobolev space $\mathrm{L}^2(0,T;\mathbf{H}^2(\Omega\backslash S(\theta_1(t),\theta_2(t))))$ [1].

This loss of regularity would harm the estimates of the nonlinear terms (see Appendix A). That is why we consider a smooth approximation $\mathbf{X}^b$ of the deformation $\mathbf{X}^a$.

In the sequel, $\mathbf{y}=(y_1,y_2)$ is the Lagrangian coordinate and $\mathbf{y}^\perp=(-y_2,y_1)$ is normal to $\mathbf{y}$. The behaviour of the smooth structure is represented in Fig. 1.1b, we give $\mathbf{X}^b$ below

$$\mathbf{X}^b(\theta_1,\theta_2,\mathbf{y}) = g_1(y_1)\mathbf{e}_{r1} + g_2(y_1)\mathbf{e}_{r2} + y_2\frac{\mathbf{N}(y_1)}{\|\mathbf{N}(y_1)\|}, \quad \mathbf{y}\in S_{\text{ref}}, \quad (\theta_1,\theta_2)\in\mathbb{D}_\Theta, \tag{1.2}$$

where g_1 and g_2 are real–valued functions. We use the notations : $\mathbf{e}_{r1}=(\cos\theta_1,\sin\theta_1)$, $\mathbf{e}_{r2}=(\cos(\theta_1+\theta_2),\sin(\theta_1+\theta_2))$, $\mathbf{N}(y_1)=g_1'(y_1)\mathbf{e}_{\theta1}+g_2'(y_1)\mathbf{e}_{\theta2}$, where $\mathbf{e}_{\theta1}=\mathbf{e}_{r1}^\perp$ and $\mathbf{e}_{\theta2}=\mathbf{e}_{r2}^\perp$. Moreover, we have $\|\mathbf{N}(y_1)\|=((N_1(y_1))^2+(N_2(y_1))^2)^{1/2}$, where N_i is the i^{th} coordinate of $\mathbf{N}$.

The function $y_1\mapsto g_1(y_1)\mathbf{e}_{r1}+g_2(y_1)\mathbf{e}_{r2}$ gives for $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$ the position of a reference curve. Every fibre of matter that is normal to this curve in the reference configuration stays normal when (θ_1,θ_2) changes. The normal direction to the curve for abscissa y_1 is given by $\mathbf{N}(y_1)$. This model is inspired from the fish–like model described in [138, Section 7].

To enforce smoothness of $\mathbf{X}^b$, g_1 and g_2 are taken as $\mathscr{C}^\infty$ functions which are smooth approximations of respectively $y_{P,1}+(y_1-y_{P,1})\chi_{[0,y_{P,1}]}(y_1)$ and $(y_1-y_{P,1})\chi_{[y_{P,1},y_{\max}]}(y_1)$, where χ_I is defined in a similar way as (1.1) for $I\subset\mathbb{R}$. For instance, let $\varepsilon>0$ and consider μ_ε a $\mathscr{C}^\infty$ cut–off function such that

$$\begin{cases} \mu_\varepsilon(y_1)=1, & \text{for } y_1<y_{P,1},\\ \mu_\varepsilon(y_1)\in[0,1], & \text{for } y_{P,1}\le y_1\le y_{P,1}+\varepsilon,\\ \mu_\varepsilon(y_1)=0, & \text{for } y_{P,1}+\varepsilon<y_1. \end{cases}$$

Then, we can use

$$\begin{cases} g_1(y_1)=y_{P,1}+\mu_\varepsilon(y_1)(y_1-y_{P,1}),\\ g_2(y_1)=(1-\mu_\varepsilon(y_1))(y_1-y_{P,1}), \end{cases}$$

in (1.2) to get a smooth deformation as in Fig. 1.1b. The velocity field of the structure is not any more discontinuous, we can thus expect the fluid to have the usual regularity of strong solutions.

Remark 1.1.1. When ε tends to 0, these functions become

$$\begin{cases} g_1(y_1)=\chi_{[a,b[}(y_1)y_1+\chi_{[b,c]}(y_1)y_{P,1},\\ g_2(y_1)=\chi_{[b,c]}(y_1)(y_1-y_{P,2}). \end{cases} \tag{1.3}$$

In this case, we recover the behaviour of a pivoting structure with two rigid solids (see Fig. 1.1c), corresponding to a transformation denoted $\mathbf{X}^c$. However, with this definition, the two solids overlap each other, so that we will not use it either in the sequel. Also let us remark that the limit $\mathbf{X}^c$ of our smooth approximation $\mathbf{X}^b$ is not the original model $\mathbf{X}^a$.

Now, let us show that we can choose $\mathbb{D}_\Theta$ such that for every $(\theta_1,\theta_2)\in\mathbb{D}_\Theta,\mathbf{X}^b(\theta_1,\theta_2,.)$ is a $\mathscr{C}^\infty$ diffeomorphism. We can compute the jacobian $\mathscr{J}_{\mathbf{X}^b}(\theta_1,\theta_2,.)$ of $\mathbf{X}^b(\theta_1,\theta_2,.)$, it fulfils $\det(\mathscr{J}_{\mathbf{X}^b}(\theta_1,\theta_2,.))=\|\mathbf{N}\|+\dfrac{y_2}{\|\mathbf{N}\|^2}\sin(\theta_2)(g_1''g_2'-g_2''g_1')$. This shows that for a given reference

1. These spaces are given here in an informal manner. They will be defined more precisely later.

configuration and for θ_2 small enough, $\det(\mathcal{J}_{\mathbf{X}^b}(\theta_1, \theta_2, .)) > 0$ everywhere. Hence this proves that $\mathbf{X}^b(\theta_1, \theta_2, .)$ is a $\mathscr{C}^\infty$ diffeomorphism for θ_2 small enough.

We shall therefore keep in mind only the example of $\mathbf{X}^b$ (see Fig. 1.1b), though our original motivation was to deal with $\mathbf{X}^a$ (see Fig. 1.1a). More generally, our approach will be applicable to many more choices of deformations $\mathbf{X}$. Let us list below the assumptions used in the sequel.

Modelling Assumptions.

- For every $\mathbf{y} \in S_{\text{ref}}$, $\mathbf{X}(0, 0, \mathbf{y}) = \mathbf{y}$. (1.4)
- $S_{\text{ref}} = S(0, 0)$ is a smooth simply connected closed subset of Ω. (1.5)
- For every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, we have $\mathbf{X}(\theta_1, \theta_2, S_{\text{ref}}) \subset \Omega$. (1.6)
- For every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1, \theta_2, .)$ is a $\mathscr{C}^\infty$ diffeomorphism (1.7)
  from S_{ref} to its image $S(\theta_1, \theta_2)$. (1.8)
- The function $\mathbf{X}$ is $\mathscr{C}^\infty$ on $\mathbb{D}_\Theta \times S_{\text{ref}}$. (1.9)
- The functions $\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .)$ and $\partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .)$ form a free family in $\mathbf{L}^2(\partial S_{\text{ref}})$ for every (θ_1, θ_2) in $\mathbb{D}_\Theta$. (1.10)

In (1.4), we have assumed that $S_{\text{ref}} = S(0, 0)$ to ease the study. Assumption (1.7) enables us to use a change of variables. This is a crucial step in our approach, as we shall see in Section 1.3.1. Assumption (1.9) has been chosen to ensure continuity of the velocity field inside the structure and on its boundary. This assumption could be weakened, as $\mathscr{C}^n$ would be sufficient for n large enough, but we keep $\mathscr{C}^\infty$ for simplicity. In our approach, Assumption (1.10) is natural and mandatory to determine the equations of the structure, as we shall see below in Section 1.1.1.2.

The inverse diffeomorphism of $\mathbf{X}(\theta_1, \theta_2, .)$ whose existence is guaranteed by (1.7) is denoted $\mathbf{Y}(\theta_1, \theta_2, .)$, we have

$$\forall(\theta_1, \theta_2) \in \mathbb{D}_\Theta, \quad \forall \mathbf{y} \in S_{\text{ref}}, \quad \mathbf{Y}(\theta_1, \theta_2, \mathbf{X}(\theta_1, \theta_2, \mathbf{y})) = \mathbf{y}. \tag{1.11}$$

The diffeomorphisms $\mathbf{X}(\theta_1, \theta_2, .)$ and $\mathbf{Y}(\theta_1, \theta_2, .)$ are illustrated in Fig. 1.2.

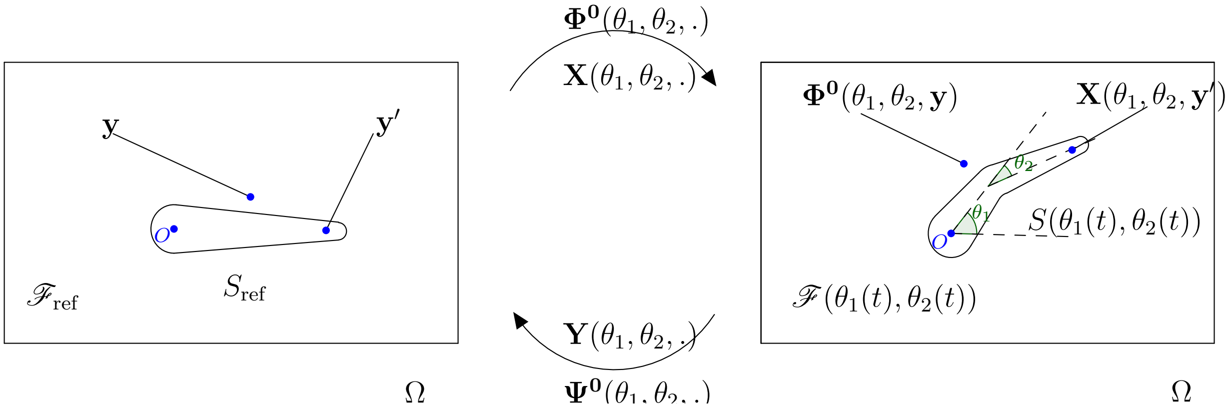


FIGURE 1.2 – Correspondance between real and reference structure configurations.

1.1.1.2 Dynamics of the structure

In order to simplify the equations of the structure, we consider the following assumption for the dynamics of the structure.

Modelling Assumption.

- No friction and no elastic energy are considered in the structure. (1.12)

The equations satisfied by the structure are obtained by a virtual work principle [22, p. 14–17]. We know that the admissible parameters of the structure are $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, and that the admissible velocities satisfy

$$\mathbf{v}_s \in \mathrm{Vect}(\partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, .), \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, .)).$$

Thus, the virtual work principle can be formulated for every time $t \in [0, T]$ as

$$\begin{cases} \text{Find } (\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta, \text{ such that} \\ \qquad\qquad \text{for every } \mathbf{w} \in \mathrm{Vect}(\partial_{\theta_1}\mathbf{X}(\theta_1(t), \theta_2(t), .), \partial_{\theta_2}\mathbf{X}(\theta_1(t), \theta_2(t), .)), \\ \displaystyle\int_{S_{\mathrm{ref}}} \rho \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y})) - \mathbf{f}_{body}(t, \mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y})) \right) \cdot \mathbf{w}(\mathbf{y}) \,\mathrm{d}\mathbf{y} \\ \qquad\qquad \displaystyle - \int_{\partial S(\theta_1(t), \theta_2(t))} \mathbf{f}_{\mathcal{F}\to S}(\gamma_x) \cdot \mathbf{w}(\mathbf{Y}(\theta_1(t), \theta_2(t), \gamma_x)) \,\mathrm{d}\gamma_x = 0, \end{cases} \tag{1.13}$$

where $\mathbf{f}_{body}$ is a distributed source term in the body (modelling for instance the gravity), ρ is a positive constant that represents the mass per unit volume of the structure in the reference configuration S_{ref} and $\mathbf{f}_{\mathcal{F}\to S}$ is the force exerted by the fluid on the structure along $\partial S(\theta_1(t), \theta_2(t))$.

Note that the presence of $\mathbf{f}_{body}$ is compatible with Assumption (1.12), as this term represents external forces. It does not depend on θ_1, θ_2 and their derivatives.

Remark 1.1.2. Assumption (1.12) has been used in (1.13) as no interior works have been considered.

Let us denote respectively $\dot{\theta}$ and $\ddot{\theta}$ the first and second time derivatives of the function θ. Then, the velocity of the structure can be written as

$$\mathbf{v}_s(t, \mathbf{y}) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y}) = \sum_{j=1}^{2} \dot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y}), \quad \forall t \in [0, T], \quad \forall \mathbf{y} \in S_{\mathrm{ref}}, \tag{1.14}$$

and its acceleration as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_s(t, \mathbf{y}) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y})) \\ &= \sum_{j=1}^{2} \ddot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y}) + \sum_{j,k=1}^{2} \dot{\theta}_j(t)\dot{\theta}_k(t)\partial_{\theta_j\theta_k}\mathbf{X}(\theta_1(t), \theta_2(t), \mathbf{y}). \end{aligned}$$

Now, problem (1.13) can be rewritten as follows

$$\begin{cases} \text{Find } (\theta_1, \theta_2) \in \mathbb{D}_\Theta, \text{ such that for } i \in \{1, 2\}, \text{ we have,} \\ \displaystyle\int_{S_{\mathrm{ref}}} \rho \sum_j \ddot{\theta}_j \partial_{\theta_j}\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) \cdot \partial_{\theta_i}\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) \,\mathrm{d}\mathbf{y} \\ \qquad\qquad \displaystyle = - \int_{S_{\mathrm{ref}}} \rho \sum_{j,k} \dot{\theta}_j\dot{\theta}_k \partial_{\theta_j\theta_k}\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) \cdot \partial_{\theta_i}\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) \,\mathrm{d}\mathbf{y} \\ \qquad\qquad \displaystyle + \int_{S_{\mathrm{ref}}} \mathbf{f}_{body}(t, \mathbf{X}(\theta_1, \theta_2, \mathbf{y})) \cdot \partial_{\theta_i}\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) \,\mathrm{d}\mathbf{y} \\ \qquad\qquad \displaystyle + \int_{\partial S(\theta_1, \theta_2)} \mathbf{f}_{\mathcal{F}\to S}(\gamma_x) \cdot \partial_{\theta_i}\mathbf{X}(\theta_1, \theta_2, \mathbf{Y}(\theta_1, \theta_2, \gamma_x)) \,\mathrm{d}\gamma_x. \end{cases}$$

Let us denote the structure body source term

$$(f_s)_i = \int_{S_{\text{ref}}} \mathbf{f}_{body}(t, \mathbf{X}(\theta_1, \theta_2, \mathbf{y})) \cdot \partial_{\theta_i}\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) \, \mathrm{d}\mathbf{y}. \tag{1.15}$$

On a matrix form, the equations of the structure read

$$\mathcal{M}_{\theta_1,\theta_2} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \mathbf{M_I}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) + \mathbf{M_A}(\theta_1, \theta_2, \mathbf{f}_{\mathcal{F}\to S}) + \mathbf{f_s} \quad \text{on } (0,T), \tag{1.16}$$

where $\mathbf{f_s} = ((f_s)_1, (f_s)_2)$ and

$$\mathcal{M}_{\theta_1,\theta_2} = \begin{pmatrix} (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S \\ (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S \end{pmatrix} \in \mathbb{R}^{2\times 2}, \tag{1.17}$$

$$\begin{aligned} &\mathbf{M_I}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ &\quad = \begin{pmatrix} -(\dot{\theta}_1^2 \partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2) + 2\dot{\theta}_1\dot{\theta}_2 \partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2) + \dot{\theta}_2^2 \partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S \\ -(\dot{\theta}_1^2 \partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2) + 2\dot{\theta}_1\dot{\theta}_2 \partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2) + \dot{\theta}_2^2 \partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S \end{pmatrix} \in \mathbb{R}^2, \end{aligned} \tag{1.18}$$

where $(.,.)_S$ is the scalar product

$$(\mathbf{\Phi}, \mathbf{\Psi})_S = \int_{S_{\text{ref}}} \rho \mathbf{\Phi}(\mathbf{y}) \cdot \mathbf{\Psi}(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \tag{1.19}$$

and

$$\mathbf{M_A}(\theta_1, \theta_2, \mathbf{f}_{\mathcal{F}\to S}) = \begin{pmatrix} \displaystyle\int_{\partial S(\theta_1,\theta_2)} \partial_{\theta_1}\mathbf{X}(\theta_1, \theta_2, \mathbf{Y}(\theta_1,\theta_2,\gamma_x)) \cdot \mathbf{f}_{\mathcal{F}\to S}(\gamma_x) \, \mathrm{d}\gamma_x \\ \displaystyle\int_{\partial S(\theta_1,\theta_2)} \partial_{\theta_2}\mathbf{X}(\theta_1, \theta_2, \mathbf{Y}(\theta_1,\theta_2,\gamma_x)) \cdot \mathbf{f}_{\mathcal{F}\to S}(\gamma_x) \, \mathrm{d}\gamma_x \end{pmatrix} \in \mathbb{R}^2. \tag{1.20}$$

The matrix $\mathcal{M}_{\theta_1,\theta_2}$ in (1.17) is the Gram matrix of the family $(\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2), \partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))$ with respect to the scalar product $(.,.)_S$. It is invertible due to Assumption (1.10) (if two $\mathscr{C}^\infty$ functions are colinear in $\mathbf{L}^2(S_{\text{ref}})$ then they are colinear in $\mathbf{L}^2(\partial S_{\text{ref}})$).

We also consider the following initial displacement and velocity for the structure

$$\begin{cases} \theta_1(0) = \theta_{1,0}, & \theta_2(0) = \theta_{2,0}, \\ \dot{\theta}_1(0) = \omega_{1,0}, & \dot{\theta}_2(0) = \omega_{2,0}. \end{cases} \tag{1.21}$$

1.1.1.3 Equations of the fluid

In our study, the global domain $\Omega = (0, L) \times (0, 1)$ represents a wind tunnel of length $L > 0$, see Fig. 1.3. Hence its boundary is composed of four regions : an inflow region $\Gamma_{\text{i}} = \{0\} \times (0, 1)$, a bottom region $\Gamma_{\text{b}} = (0, L) \times \{0\}$, a top region $\Gamma_{\text{t}} = (0, L) \times \{1\}$ and an outflow region $\Gamma_{\text{N}} = \{L\} \times (0, 1)$. We denote $\Gamma_{\text{w}} = \Gamma_{\text{t}} \cup \Gamma_{\text{b}}$ the part of the boundary corresponding to walls and $\Gamma_{\text{D}} = \Gamma_{\text{i}} \cup \Gamma_{\text{w}}$ the part of the boundary where Dirichlet conditions are imposed.

At time t, the structure occupies the volume $S(\theta_1(t), \theta_2(t))$, therefore the fluid fills the domain $\mathscr{F}(\theta_1(t), \theta_2(t)) = \Omega \backslash S(\theta_1(t), \theta_2(t))$.

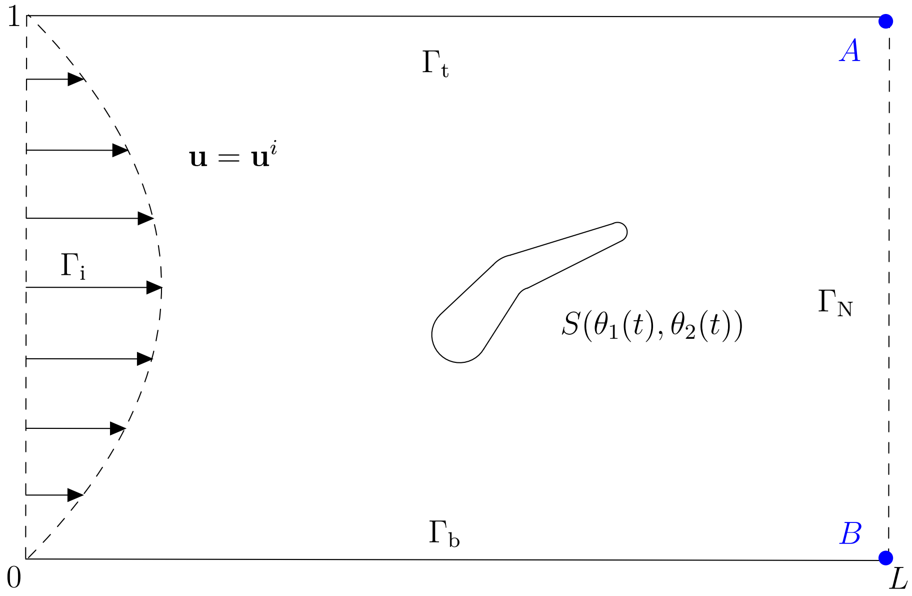


FIGURE 1.3 – The geometrical configuration.

The velocity of the fluid is modelled by the incompressible Navier–Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}\cdot\nabla)\mathbf{u}(t,\mathbf{x}) - \operatorname{div}\sigma_F(\mathbf{u}(t,\mathbf{x}), p(t,\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,T), \ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\ \operatorname{div}\,\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,T), \ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in(0,T), \ \mathbf{x}\in\Gamma_{\rm i}, \\ \mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,T), \ \mathbf{x}\in\Gamma_{\rm w}, \\ \sigma_F(\mathbf{u}(t,\mathbf{x}), p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t\in(0,T), \ \mathbf{x}\in\Gamma_{\rm N}, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}), \end{cases} \tag{1.22}$$

where $\mathbf{u}(t,\mathbf{x})$ and $p(t,\mathbf{x})$ are velocity and pressure of the fluid at point $\mathbf{x}$ and time t, and

$$\sigma_F(\mathbf{u},p) = \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - p\mathrm{I},$$

is the stress tensor of the fluid, where $\nu > 0$ is the kinematic viscosity of the fluid. The vector $\mathbf{n}$ denotes the unitary outward normal to Ω. The term $\mathbf{f}_{\mathscr{F}}(t,\mathbf{x})$ in $(1.22)_1$ is a force per unit mass exerted on the fluid. Moreover, a nonhomogeneous Dirichlet boundary condition $\mathbf{u}^i$ is imposed on the inflow region $\Gamma_{\rm i}$ and we consider an initial condition $\mathbf{u}_0$ for the fluid velocity. Of course, these equations should be completed with suitable boundary conditions on $\partial S(\theta_1(t),\theta_2(t))$ that will be made precise in Section 1.1.1.4.

1.1.1.4 Interface between the fluid and the structure

The velocity $\mathbf{u}$ of the fluid fulfils an adherence condition with the boundary of the structure whose velocity is given in (1.14),

$$\mathbf{u}(t,\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y})) = \sum_{j=1}^{2}\dot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{y}), \quad t\in(0,T), \quad \mathbf{y}\in\partial S_{\rm ref}.$$

Note that this no–slip boundary condition corresponds to the continuity of the velocity through the interface between the fluid and the structure and can also be rewritten as

$$\mathbf{u}(t,\mathbf{x}) = \sum_{j=1}^{2}\dot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), \quad t\in(0,T), \quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)). \quad (1.23)$$

The forces exerted by the fluid on the structure are given by the stress tensor of the fluid

$$\mathbf{f}_{\mathcal{F}\to S} = -\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}, \quad t\in(0,T), \quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)), \quad (1.24)$$

where $\mathbf{n}_{\theta_1,\theta_2}(\mathbf{x})$ is the outward unitary normal to the fluid domain $\mathscr{F}(\theta_1(t),\theta_2(t))$ on $\partial S(\theta_1(t),\theta_2(t))$.

1.1.1.5 The complete set of equations

The full set of equations is given by (1.16), (1.21), (1.22), (1.23) and (1.24). Note that the coupling between the fluid and the structure appears in equations (1.22) (as the fluid domain depends on θ_1 and θ_2), (1.23) and (1.24).

The final considered system is given by the following set of equations

$$\begin{cases}
\dfrac{\partial\mathbf{u}}{\partial t}(t,\mathbf{x})+(\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x})-\operatorname{div}\,\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))=\mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,T), \quad \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\operatorname{div}\,\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,T), \quad \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x}) = \displaystyle\sum_{j=1}^{2}\dot{\theta}_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,T), \quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in(0,T), \quad \mathbf{x}\in\Gamma_{\mathrm{i}},\\
\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,T), \quad \mathbf{x}\in\Gamma_{\mathrm{w}},\\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t\in(0,T), \quad \mathbf{x}\in\Gamma_{\mathrm{N}},\\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}),\\
\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}=\mathbf{M_I}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) & \\
\qquad\qquad +\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2})+\mathbf{f_s}, & t\in(0,T),\\
\theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, & \\
\dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}. &
\end{cases} \quad (1.25)$$

1.1.2 Statement of the main result

In this section, after setting up the functional framework, we present our existence result for solutions to problem (1.25). In the sequel, $\mathscr{F}_0 = \mathscr{F}(\theta_{1,0},\theta_{2,0})$ denotes the initial fluid domain and $S_0 = S(\theta_{1,0},\theta_{2,0})$ the initial configuration of the structure. For the sake of simplicity, the initial displacement of the structure is taken equal to zero,

$$\theta_{1,0} = \theta_{2,0} = 0.$$

This can be done without loss of generality by the change of variables

$$(\theta_1,\theta_2)\mapsto(\theta_1-\theta_{1,0},\theta_2-\theta_{2,0}).$$

Moreover, the reference configuration for the structure S_{ref} and for the fluid $\mathscr{F}_{\mathrm{ref}}$ are taken as the initial configuration,

$$S_{\mathrm{ref}} = S_0 = S(0,0), \ \mathscr{F}_{\mathrm{ref}} = \mathscr{F}_0 = \mathscr{F}(0,0).$$

1.1.2.1 Functional spaces

Sobolev spaces. In the sequel, $\mathrm{H}^s(\mathscr{F}_0)$ is the usual Sobolev space of order $s \geq 0$. We identify $\mathrm{L}^2(\mathscr{F}_0)$ with $\mathrm{H}^0(\mathscr{F}_0)$. We will denote $\mathbf{L}^2(\mathscr{F}_0) = (\mathrm{L}^2(\mathscr{F}_0))^2$, $\mathbf{H}^s(\mathscr{F}_0) = (\mathrm{H}^s(\mathscr{F}_0))^2$ and so on.

Corners issues. The domain considered for the fluid has four corners of angle $\pi/2$. The ones that are located between Dirichlet and Neumann boundary conditions induce singularities, we denote them $A = (L, 1)$ and $B = (L, 0)$ (see Fig. 1.3). We also denote $\mathscr{J}_{d,n} = \{A, B\}$ the set of these corners and we define the distance of a point $\mathbf{x}$ from these corners

$$\text{for } j \in \mathscr{J}_{d,n}, \quad \text{for } \mathbf{x} \in \Omega, \quad r_j(\mathbf{x}) = \mathrm{d}(\mathbf{x}, j). \tag{1.26}$$

Note that corners between two Dirichlet boundary conditions do not induce singularities as soon as suitable compatibility conditions are satisfied. We report to [111] for more details.

Weighted Sobolev spaces. The strong solution to the Stokes problem in the domain with corners A and B and with a source term in $\mathbf{L}^2(\mathscr{F}_0)$ belongs to a classical Sobolev space of lower order than what we usually have with smooth domains. In order to get the usual gain of regularity between solutions and source terms, we have to study the solution in adapted Sobolev spaces. As the loss of regularity is located around corners A and B, we can recover the usual regularity if we consider norms that are suitably weighted near these corners. The weighted Sobolev spaces are then defined for $\beta > 0$ as

$$\begin{aligned} \mathbf{H}^2_\beta(\mathscr{F}_0) &= \{\mathbf{u} \ \text{ with } \ \|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_0)} < +\infty\}, \\ \mathrm{H}^1_\beta(\mathscr{F}_0) &= \{p \ \text{ with } \ \|p\|_{\mathrm{H}^1_\beta(\mathscr{F}_0)} < +\infty\}, \end{aligned}$$

where the norms $\|.\|_{\mathbf{H}^2_\beta(\mathscr{F}_0)}$ and $\|.\|_{\mathrm{H}^1_\beta(\mathscr{F}_0)}$ are given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\beta(\mathscr{F}_0)} = \sum_{|\alpha|=0}^{2} \sum_{i=1}^{2} \int_{\mathscr{F}_0} \left(\prod_{j \in \mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y}) \right) |\partial^\alpha u_i(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}, \tag{1.27}$$

and

$$\|p\|^2_{\mathrm{H}^1_\beta(\mathscr{F}_0)} = \sum_{|\alpha|=0}^{1} \int_{\mathscr{F}_0} \left(\prod_{j \in \mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y}) \right) |\partial^\alpha p(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}. \tag{1.28}$$

Here the sum is on all multi–index α of length $|\alpha| \leq 2$ for (1.27) and $|\alpha| \leq 1$ for (1.28) and r_j is defined in (1.26).

Steady Stokes problem with corners. Let us denote $\mathbf{n}_0$ the outward unitary normal to $\mathscr{F}_0$ on ∂S_0. The following lemma from [118] explains how and why the spaces $\mathbf{H}^2_\beta$ and H^1_β appear in the context of corners. It gives the result expected for the steady Stokes problem in $\mathscr{F}_0$ with weigthed Sobolev spaces and the regularity obtained in the classical Sobolev spaces.

Lemma 1.1.3. *[118, Theorem 2.5.] Let us assume that $\mathbf{f}_{\mathscr{F}} \in \mathbf{L}^2(\mathscr{F}_0)$. The unique solution $(\mathbf{u}, p)$ to the Stokes problem*

$$\begin{cases} -\mathrm{div}\ \sigma_F(\mathbf{u}, p) = \mathbf{f}_{\mathscr{F}} & \text{in } \mathscr{F}_0, \\ \mathrm{div}\ \mathbf{u} = 0 & \text{in } \mathscr{F}_0, \\ \mathbf{u} = 0 & \text{on } \Gamma_{\mathrm{D}} \cup \partial S_0, \\ \sigma_F(\mathbf{u}, p)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{1.29}$$

belongs to $\mathbf{H}^2_\beta(\mathscr{F}_0)\times \mathrm{H}^1_\beta(\mathscr{F}_0)$ *for some* $\beta \in (0,1/2)$ *and to* $\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0)\times \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0)$ *for some* $\varepsilon_0 \in (0,1/2)$. *Moreover, we have the following estimate*

$$\|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_0)\cap \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0)} + \|p\|_{\mathrm{H}^1_\beta(\mathscr{F}_0)\cap \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0)} \leq C\|\mathbf{f}_{\mathscr{F}}\|_{\mathbf{L}^2(\mathscr{F}_0)}. \tag{1.30}$$

Note that the regularity proven in Lemma 1.1.3 gives a meaning to all integrations by parts as $p_{|\partial\mathscr{F}_0}$ and $\partial_{\mathbf{n}_0}\mathbf{u}_{|\partial\mathscr{F}_0}$ are well defined traces for $(\mathbf{u},p)\in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0)\times \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0)$.

Also note that according to [80, Theorem 1.4.3.1], there exists a continuous extension operator from $\mathbf{H}^s(\mathscr{F}_0)$ to $\mathbf{H}^s(\mathbb{R}^2)$ for every $s>0$. This implies that all the classical Sobolev injections and interpolations are valid despite the presence of corners as they can be led in $\mathbb{R}^2$.

1.1.2.2 Local existence of a strong solution to the problem

The diffeomorphism $\mathbf{\Phi^0}$. A classical difficulty in fluid–structure problems is that the fluid domain changes over time. The classical way to get rid of this difficulty is to use a change of variables on $\mathbf{u}$ and p in order to bring the study back into a fixed domain. This procedure uses a diffeomorphism that we have to define properly. When the state of the structure depends only on a finite number of parameters, it is convenient to construct this diffeomorphism as an extension of the deformation of the structure into the fluid domain.

The diffeomorphism used is defined as an extension of the diffeomorphism $\mathbf{X}$ given for the structure. Hence, we need the extension operator defined below.

Lemma 1.1.4. *There exists a linear extension operator* $\mathcal{E} : \mathbf{W}^{3,\infty}(S_0) \to \mathbf{W}^{3,\infty}(\Omega)\cap \mathbf{H}^1_0(\Omega)$ *such that for every* $\mathbf{w}\in \mathbf{W}^{3,\infty}(S_0)$

- (i) $\mathcal{E}(\mathbf{w}) = \mathbf{w}$ in S_0,
- (ii) $\mathcal{E}(\mathbf{w})$ has support within $\Omega_\varepsilon = \{\mathbf{x}\in\Omega \mid \mathrm{d}(x,\partial\Omega)>\varepsilon\}$ for some $\varepsilon>0$ such that $\mathrm{d}(S(\theta_1,\theta_2),\partial\Omega)>2\varepsilon$ for all $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$,
- (iii) $\|\mathbf{w}\|_{\mathbf{W}^{3,\infty}(\Omega)} \leq C\|\mathbf{w}\|_{\mathbf{W}^{3,\infty}(S_0)}$, for some $C>0$.

Proof. Extension results are classical, we can for instance find an extension result for smooth domains in [104, Lemma 12.2]. We can get the result by multiplying the extension function of [104, Lemma 12.2] by a cut–off function in $\mathcal{D}(\Omega_\varepsilon)$. □

Then we define the following function

$$\mathbf{\Phi^0}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y} + \mathcal{E}\Big(\mathbf{X}(\theta_1,\theta_2,.) - \mathrm{Id}\Big)(\mathbf{y}), \quad \forall(\theta_1,\theta_2)\in\mathbb{D}_\Theta, \quad \forall \mathbf{y}\in\Omega, \tag{1.31}$$

where Id denotes the identity function.

We have $\nabla\mathbf{\Phi^0}(0,0,\mathbf{y}) = \mathrm{I}$ for every $\mathbf{y}\in\Omega$, hence $\det(\nabla\mathbf{\Phi^0}(0,0,\mathbf{y}))=1$. Then for every $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$ small enough, the function $\mathbf{\Phi^0}(\theta_1,\theta_2,.)$ is a diffeomorphism close to the identity function. We denote $\mathbf{\Psi^0}(\theta_1,\theta_2,.)$ the inverse diffeomorphism of $\mathbf{\Phi^0}(\theta_1,\theta_2,.)$

$$\forall(\theta_1,\theta_2)\in\mathbb{D}_\Theta, \quad \forall\mathbf{y}\in\Omega, \quad \mathbf{\Psi^0}(\theta_1,\theta_2,\mathbf{\Phi^0}(\theta_1,\theta_2,\mathbf{y})) = \mathbf{y}. \tag{1.32}$$

We can prove that $\mathbf{\Phi^0}$ and $\mathbf{\Psi^0}$ belong to $\mathscr{C}^\infty(\mathbb{D}_\Theta,\mathbf{W}^{3,\infty}(\Omega))$. These diffeomorphisms are represented in Fig. 1.2.

The properties of $\mathcal{E}$ imply that

$$\text{for every } (\theta_1,\theta_2)\in\mathbb{D}_\Theta,\ \ \mathbf{\Phi^0}(\theta_1,\theta_2,S_0) = S(\theta_1,\theta_2) \quad \text{and} \quad \forall\mathbf{y}\in\Omega\backslash\Omega_\varepsilon,\ \ \mathbf{\Phi^0}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y}. \tag{1.33}$$

The inflow boundary conditions. We use the following space to define the admissible boundary conditions on the inflow part of the boundary Γ_{i},

$$\mathbf{U}^i = \left\{ \begin{array}{ll} \mathbf{u}^i \in \mathbf{H}^{3/2}(\Gamma_{\mathrm{i}}) \ \text{ with } \ \mathbf{u}^i_{|\partial\Gamma_{\mathrm{i}}} = 0, & \displaystyle\int_0^{1/4} \frac{|\partial_{y_2} u_2^i(y_2)|^2}{y_2} \mathrm{d}y_2 < +\infty, \\ & \displaystyle\int_{3/4}^1 \frac{|\partial_{y_2} u_2^i(y_2)|^2}{1-y_2} \mathrm{d}y_2 < +\infty \end{array} \right\}. \tag{1.34}$$

The conditions with integrals in the definition of $\mathbf{U}^i$ are chosen to match the homogeneous boundary conditions on Γ_{w}. We now state the following existence theorem.

Theorem 1.1.5 (Local existence in time of a solution)**.** *Let $T_0 > 0$, let $\mathbf{u}^i \in \mathrm{H}^1(0,T_0;\mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0},\omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions*

$$\begin{cases} \operatorname{div} \mathbf{u}_0 = 0 & \text{in } \mathscr{F}_0, \\ \mathbf{u}_0(.) = \displaystyle\sum_{j=1}^2 \omega_{j,0} \partial_{\theta_j} \mathbf{X}(0,0,.) & \text{on } \partial S_0, \\ \mathbf{u}_0 = \mathbf{u}^i(0,.) & \text{on } \Gamma_{\mathrm{i}}, \\ \mathbf{u}_0 = 0 & \text{on } \Gamma_{\mathrm{w}}. \end{cases} \tag{1.35}$$

Let $\mathbf{f}_{\mathscr{F}} \in \mathrm{L}^2(0,T_0;\mathbf{W}^{1,\infty}(\Omega))$ and $\mathbf{f_s} \in \mathrm{L}^2(0,T_0;\mathbb{R}^2)$. Then there exists a time $T \in (0,T_0]$ such that problem (1.25) *admits a unique solution $(\mathbf{u},p,\theta_1,\theta_2)$ with the following regularity*

$$\begin{array}{l} (\theta_1,\theta_2) \in \mathrm{H}^2(0,T;\mathbb{D}_\Theta), \\ \mathbf{u}(t,\mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y})) \in \mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0)), \\ p(t,\mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y})) \in \mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0)). \end{array}$$

Moreover, we have the estimate

$$\begin{aligned} &\|\mathbf{u}(t,\mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y}))\|_{\mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))\cap\mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0))\cap\mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} \\ &\quad + \|p(t,\mathbf{\Phi^0}(\theta_1(t),\theta_2(t),\mathbf{y}))\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0))} + \|(\theta_1,\theta_2)\|_{\mathrm{H}^2(0,T;\mathbb{D}_\Theta)} \\ &\qquad \le C\big(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_{\mathscr{F}}\|_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{H}^{3/2}(\Gamma_{\mathrm{i}}))} + \|\mathbf{f_s}\|_{\mathrm{L}^2(0,T_0;\mathbb{R}^2)}\big). \end{aligned}$$

The proof of Theorem 1.1.5 mainly follows the one in [34] and is presented in Section 1.3.3.

1.1.3 Scientific context

Existence of strong solutions to fluid–structure problems is already available for several cases. For instance the problems of a fluid coupled with rigid bodies [79, 147, 148], a plate [133] or a beam [102, 116] have already been investigated.

Existence of a weak solution has also been proven for a fluid coupled with a plate [47].

In the current study, we focus on a deformable structure depending on a finite number of parameters. A close situation has already been investigated for a finite dimensional approximation of a plate [34], where the functions $\partial_{\theta_j}\mathbf{X}$ fulfil a relation mandatory to ensure the null divergence of the fluid.

In contrast to [34], the case considered in the current paper deals with an intrinsically finite dimensional structure. Hence, the functions $\partial_{\theta_j}\mathbf{X}$ do not fulfil such a relation and some parts of the proof in [34] have then to be modified.

Additional difficulties are induced by the corners on $\partial\Omega$, more information can be found in [111, 118].

1.1.4 Outline of the paper

In Section 1.2, we study the linearized problem in the fixed domain $\mathscr{F}_0$. We prove existence of strong solutions to this linearized problem. Then, in Section 1.3, we prove local existence of solutions to the nonlinear system. The proof of the estimates of the nonlinear terms can be found in Appendix A.

1.2 Existence of solution to the linearized problem

In this section we study the linearization of the problem (1.25), first with only source terms $\mathbf{f}$ and $\mathbf{s}$ and then with all source terms and boundary data. These equations are written in the fixed domain $\mathscr{F}_0$ using a change of variables explained in Section 1.3.1. In the sequel, $(\tilde{\mathbf{u}}, \tilde{p})$ denotes the velocity and the pressure of the fluid in the fixed domain $\mathscr{F}_0$. We denote $T > 0$ the considered final time.

1.2.1 Linearized problem with nonhomogeneous source terms

Let us study the following problem

$$
\begin{cases}
\dfrac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu \Delta \tilde{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} & \text{in } (0,T)\times\mathscr{F}_0,\\
\operatorname{div} \tilde{\mathbf{u}} = 0 & \text{in } (0,T)\times\mathscr{F}_0,\\
\tilde{\mathbf{u}} = \dot{\theta}_1 \partial_{\theta_1}\mathbf{\Phi^0}(0,0,.) + \dot{\theta}_2 \partial_{\theta_2}\mathbf{\Phi^0}(0,0,.) & \text{on } (0,T)\times\partial S_0,\\
\tilde{\mathbf{u}} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{i}},\\
\tilde{\mathbf{u}} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{w}},\\
\sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}},\\
\tilde{\mathbf{u}}(0,.) = \mathbf{u}_0(.) & \text{in } \mathscr{F}_0,\\
\mathcal{M}_{0,0}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0 \cdot \partial_{\theta_1}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0 \cdot \partial_{\theta_2}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y \end{pmatrix} + \mathbf{s} & \text{on } (0,T),\\
\theta_1(0) = 0, \qquad \theta_2(0) = 0, \\
\dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0},
\end{cases}
\tag{1.36}
$$

where the unkwnows are $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ and the source terms are $(\mathbf{f}, \mathbf{s}) \in \mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0)) \times \mathrm{L}^2(0,T;\mathbb{R}^2)$. We will show later that this system corresponds to the linearization of the nonlinear problem (1.25) transported in the fixed initial configuration $\mathscr{F}_0$.

Remark 1.2.1. The state $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ of problem (1.36) can be reduced to $(\tilde{\mathbf{u}}, \tilde{p}, \dot{\theta}_1, \dot{\theta}_2)$. Considering the velocity of the structure instead of its position is sufficient to solve (1.36). However, we prefer to consider the full state $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$, as it is useful in the sequel to deal with the nonlinear problem.

In the sequel, the following spaces are used

$$\mathbb{U}_T = \mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0)), \tag{1.37}$$

$$\mathbb{P}_T = \mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0)), \tag{1.38}$$

$$\Theta_T = \mathrm{H}^2(0,T;\mathbb{R}^2), \tag{1.39}$$

$$\mathbb{F}_T = \mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0)), \tag{1.40}$$

$$\mathbb{G}_T = \mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial S_0)), \tag{1.41}$$

$$\mathbb{S}_T = \mathrm{L}^2(0,T;\mathbb{R}^2). \tag{1.42}$$

We endow Θ_T with the following norm

$$\|(\theta_1,\theta_2)\|_{\Theta_T} = \|(\theta_1,\theta_2)\|_{\mathrm{H}^2(0,T)} + \|(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,T)} + \|(\dot\theta_1,\dot\theta_2)\|_{\mathrm{L}^\infty(0,T)},$$

the other spaces are endowed with their natural norms. The norm $\|.\|_{\Theta_T}$ has been chosen so that we have the estimate $\|(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,T)} + \|(\dot\theta_1,\dot\theta_2)\|_{\mathrm{L}^\infty(0,T)} \leq C\|(\theta_1,\theta_2)\|_{\Theta_T}$ where C does not depend on T. Note that with the natural norm of Θ_T, C would depend on T.

Let us fix an arbitrary time $T_0 > 0$, e.g. $T_0 = 1$. We want to prove the following result.

Proposition 1.2.2. *There exists a constant $C > 0$ such that for all $T \in (0,T_0)$, C does not depend on T, for all $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0},\omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions* (1.35) *(with $\mathbf{u}^i = 0$) and every $(\mathbf{f},\mathbf{s}) \in \mathbb{F}_T \times \mathbb{S}_T$, problem* (1.36) *admits a unique solution*

$$(\tilde{\mathbf{u}},\tilde p,\theta_1,\theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T.$$

Moreover, the following estimate holds

$$\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde p\|_{\mathbb{P}_T} + \|(\theta_1,\theta_2)\|_{\Theta_T} \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}_T} + \|\mathbf{s}\|_{\mathbb{S}_T}). \tag{1.43}$$

In order to prove Proposition 1.2.2, we will study the problem (1.36) under its semigroup formulation. Let us define the space

$$\mathbb{H} = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}},\theta_1,\theta_2,\omega_1,\omega_2) \in \mathbf{L}^2(\mathscr{F}_0)\times\mathbb{R}^4, & \mathrm{div}\,\tilde{\mathbf{u}} = 0 \text{ in } \mathscr{F}_0, \quad \tilde{\mathbf{u}}\cdot\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{D}}, \\ & \tilde{\mathbf{u}}\cdot\mathbf{n}_0 = \sum\limits_j \omega_j \partial_{\theta_j}\mathbf{\Phi^0}(0,0,.)\cdot\mathbf{n}_0 \text{ on } \partial S_0 \end{array} \right\}, \tag{1.44}$$

where $\mathbf{n}_0$ is the unitary outward normal to the fluid domain $\mathscr{F}_0$. The space $\mathbb{H}$ is endowed with the scalar product $(.,.)_{\mathbb{H}}$ of $\mathbf{L}^2(\mathscr{F}_0)\times\mathbb{R}^4$ defined by

$$\Big((\tilde{\mathbf{u}}^a,\theta_1^a,\theta_2^a,\omega_1^a,\omega_2^a),(\tilde{\mathbf{u}}^b,\theta_1^b,\theta_2^b,\omega_1^b,\omega_2^b)\Big)_{\mathbb{H}} = \int_{\mathscr{F}_0} \tilde{\mathbf{u}}^a\cdot\tilde{\mathbf{u}}^b \,\mathrm{d}\mathbf{y} + (\theta_1^a\;\;\theta_2^a)\begin{pmatrix}\theta_1^b\\ \theta_2^b\end{pmatrix} + (\omega_1^a\;\;\omega_2^a)\,\mathcal{M}_{0,0}\begin{pmatrix}\omega_1^b\\ \omega_2^b\end{pmatrix}.$$

We also define

$$\mathbb{V} = \left\{ \begin{array}{ll} (\tilde{\mathbf{u}},\theta_1,\theta_2,\omega_1,\omega_2) \in \mathbf{H}^1(\mathscr{F}_0)\times\mathbb{R}^4, & \mathrm{div}\,\tilde{\mathbf{u}} = 0 \text{ in } \mathscr{F}_0, \quad \tilde{\mathbf{u}} = 0 \text{ on } \Gamma_{\mathrm{D}}, \\ & \tilde{\mathbf{u}} = \sum\limits_j \omega_j \partial_{\theta_j}\mathbf{\Phi^0}(0,0,.) \text{ on } \partial S_0 \end{array} \right\}, \tag{1.45}$$

endowed with the scalar product $(.,.)_{\mathbb{V}}$ of $\mathbf{H}^1(\mathscr{F}_0)\times\mathbb{R}^4$ defined by

$$\begin{aligned}\Big((\tilde{\mathbf{u}}^a,\theta_1^a,\theta_2^a,\omega_1^a,\omega_2^a),(\tilde{\mathbf{u}}^b,\theta_1^b,\theta_2^b,\omega_1^b,\omega_2^b)\Big)_{\mathbb{V}} =& \int_{\mathscr{F}_0} (\tilde{\mathbf{u}}^a\cdot\tilde{\mathbf{u}}^b + \nabla\tilde{\mathbf{u}}^a : \nabla\tilde{\mathbf{u}}^b)\,\mathrm{d}\mathbf{y} \\ &+ (\theta_1^a\;\;\theta_2^a)\begin{pmatrix}\theta_1^b\\ \theta_2^b\end{pmatrix} + (\omega_1^a\;\;\omega_2^a)\,\mathcal{M}_{0,0}\begin{pmatrix}\omega_1^b\\ \omega_2^b\end{pmatrix}.\end{aligned}$$

In the sequel, we denote $(f_j)_{j=1,2} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Lemma 1.2.3. *The orthogonal space to* $\mathbb{H}$ *with respect to the scalar product* $(.,.)_{\mathbb{H}}$ *is*

$$(\mathbb{H})^{\perp} = \left\{ \left(\nabla p, 0, 0, -\mathcal{M}_{0,0}^{-1} \Big(\int_{\partial S_0} p \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi^0}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \Big)_{j=1,2} \right) \text{ with } p \in \mathrm{H}^1(\mathscr{F}_0) \ , \ p = 0 \text{ on } \Gamma_{\mathrm{N}} \right\},$$

where $\mathcal{M}_{0,0}$ *is defined in* (1.17)*.*

Proof. Let $(\tilde{\mathbf{u}}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a) \in \mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ such that for every $(\tilde{\mathbf{u}}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \in \mathbb{H}$,

$$\Big((\tilde{\mathbf{u}}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a) , (\tilde{\mathbf{u}}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \Big)_{\mathbb{H}} = 0.$$

By taking $\tilde{\mathbf{u}}^b = 0$ and $\omega_1^b = \omega_2^b = 0$, we easily obtain $\theta_1^a = \theta_2^a = 0$. With $\omega_1^b = \omega_2^b = 0$, we also get

$$\int_{\mathscr{F}_0} \tilde{\mathbf{u}}^a \cdot \tilde{\mathbf{u}}^b \, \mathrm{d}\mathbf{y} = 0, \quad \forall \tilde{\mathbf{u}}^b \in \mathbf{L}^2(\mathscr{F}_0) \text{ such that } \mathrm{div}\, \tilde{\mathbf{u}}^b = 0 \text{ in } \mathscr{F}_0 \text{ and } \tilde{\mathbf{u}}^b \cdot \mathbf{n}_0 = 0 \text{ on } \Gamma_{\mathrm{D}} \cup \partial S_0,$$

which implies, according to [118, Lemma 2.2], $\tilde{\mathbf{u}}^a = \nabla p$, where $p \in \mathrm{H}^1(\mathscr{F}_0)$ and $p = 0$ on Γ_{N}. Now,

$$\int_{\mathscr{F}_0} \nabla p \cdot \tilde{\mathbf{u}}^b \, \mathrm{d}\mathbf{y} + \sum_{j,k} \omega_j^a \omega_k^b (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S = 0,$$

becomes with the divergence formula and the compatibility condition in (1.44)

$$\sum_j \omega_j^b \int_{\partial S_0} p \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi^0}(0,0,\gamma_y) \, \mathrm{d}\gamma_y + \sum_{j,k} \omega_j^a \omega_k^b (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S = 0,$$

then

$$\int_{\partial S_0} p \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi^0}(0,0,\gamma_y) \, \mathrm{d}\gamma_y + \sum_{k} \omega_k^a (\partial_{\theta_j} \mathbf{X}(0,0,.), \partial_{\theta_k} \mathbf{X}(0,0,.))_S = 0,$$

which yields a first inclusion. The converse inclusion is obtained via an integration by parts. □

We define the operator $(A, D(A))$ on $\mathbb{H}$ as

$$D(A) = \left\{ \begin{array}{r} (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, \tilde{\mathbf{u}} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0), \exists \tilde{p} \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0) \text{ such that} \\ \mathrm{div}\, \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \in \mathbf{L}^2(\mathscr{F}_0) \text{ and } \sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\}, \quad (1.46)$$

where ε_0 is introduced in Lemma 1.1.3, and

$$A \begin{pmatrix} \tilde{\mathbf{u}} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \mathrm{div}\, \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \Big(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi^0}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \Big)_{j=1,2} \end{pmatrix}, \quad (1.47)$$

where $\Pi_{\mathbb{H}}$ is the orthogonal projector from $\mathbf{L}^2(\mathscr{F}_0) \times \mathbb{R}^4$ onto $\mathbb{H}$ with respect to $(.,.)_{\mathbb{H}}$.

Lemma 1.2.4. *The operator* A *is uniquely defined.*

Proof. Let $(\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in D(A)$ and consider two functions p, $q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0)$ satisfying the conditions appearing into the definition of $D(A)$. Then, div $\sigma_F(0, p-q) = -\nabla(p-q) \in \mathbf{L}^2(\mathscr{F}_0)$ implies $p - q \in \mathrm{H}^1(\mathscr{F}_0)$, and $\sigma_F(0, p-q)\mathbf{n} = 0$ on Γ_{N} implies $p - q = 0$ on Γ_{N}.

Now,

$$\begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, p) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, p)\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix} - \begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, q) \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, q)\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix}$$
$$= \begin{pmatrix} \nabla(p-q) \\ 0 \\ 0 \\ -\mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_0} (p-q)\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix},$$

which belongs to $\mathbb{H}^\perp$ according to Lemma 1.2.3. Therefore A is uniquely defined. □

We define the bilinear form a_1 on $\mathbb{V} \times \mathbb{V}$ for every $(\tilde{\mathbf{u}}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a)$ and $(\tilde{\mathbf{u}}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)$ in $\mathbb{V}$ by

$$\mathrm{a}_1((\tilde{\mathbf{u}}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a), (\tilde{\mathbf{u}}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)) = \frac{\nu}{2}\int_{\mathscr{F}_0} (\nabla\tilde{\mathbf{u}}^a + (\nabla\tilde{\mathbf{u}}^a)^T) : (\nabla\tilde{\mathbf{u}}^b + (\nabla\tilde{\mathbf{u}}^b)^T)\,\mathrm{d}\mathbf{y}.$$

We define the operator $(A_1, D(A))$ on $\mathbb{H}$ by

$$D(A_1) = \{\mathbf{z} \in \mathbb{V} \ \text{ with } \ \tilde{\mathbf{z}} \mapsto \mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) \text{ is } \mathbb{H} - \text{continuous }\},$$

and

$$\forall \mathbf{z} \in D(A_1), \quad \forall \tilde{\mathbf{z}} \in \mathbb{V}, \quad (A_1\mathbf{z}, \tilde{\mathbf{z}})_{\mathbb{H}} = -\mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}).$$

Lemma 1.2.5. *We have*

$$D(A_1) = D(A),$$

and

$$A_1 \begin{pmatrix} \tilde{\mathbf{u}} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix}.$$

Proof. The inclusion $D(A) \subset D(A_1)$ comes easily. Moreover, for every $\mathbf{z} = (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in D(A)$, an integration by parts yields

$$\forall \tilde{\mathbf{z}} \in \mathbb{V}, \quad (A_1\mathbf{z}, \tilde{\mathbf{z}})_{\mathbb{H}} = \left(\begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \tilde{p}) \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, \tilde{p})\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix}, \tilde{\mathbf{z}}\right)_{\mathbb{H}}.$$

Let us now prove the reverse inclusion $D(A_1) \subset D(A)$. Let $\mathbf{z} = (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in D(A_1)$. According to Riesz representation theorem, there exists $\mathbf{f} \in \mathbb{H}$ such that

$$\forall \tilde{\mathbf{z}} \in \mathbb{V}, \quad \mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) = (\mathbf{f}, \tilde{\mathbf{z}})_{\mathbb{H}}.$$

We write $\mathbf{f} = (\mathbf{f}_{\mathbf{u}}, \mathbf{f}_{\theta_1}, \mathbf{f}_{\theta_2}, \mathbf{f}_{\omega_1}, \mathbf{f}_{\omega_2})$. For $\tilde{\mathbf{v}} \in \mathcal{D}_{\mathrm{div}} = \{\mathbf{u} \in (\mathscr{C}_c^\infty(\mathscr{F}_0))^2 \text{ with } \operatorname{div} \mathbf{u} = 0\}$, we know that $(\tilde{\mathbf{v}}, 0, 0, 0, 0)$ belongs to $\mathbb{V}$, and an integration by parts yields

$$\mathrm{a}_1(\mathbf{z}, (\tilde{\mathbf{v}}, 0, 0, 0, 0)) = \int_{\mathscr{F}_0} (-\operatorname{div} \sigma_F(\tilde{\mathbf{u}}, 0)) \cdot \tilde{\mathbf{v}} \, \mathrm{d}\mathbf{y} = \int_{\mathscr{F}_0} \mathbf{f}_{\mathbf{u}} \cdot \tilde{\mathbf{v}} \, \mathrm{d}\mathbf{y}.$$

Then, according to [146, Lemma 2.2.2], there exists $\widehat{q} \in \mathrm{L}^2(\mathscr{F}_0)$ such that

$$-\operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \widehat{q}) = \mathbf{f}_{\mathbf{u}} \text{ in } \mathscr{F}_0, \tag{1.48}$$

and thus $\operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \widehat{q})$ belongs to $\mathbf{L}^2(\mathscr{F}_0)$, which gives a meaning to $\sigma_F(\tilde{\mathbf{u}}, \widehat{q})\mathbf{n}_0$ on $\partial\mathscr{F}_0$.

Now, let us prove that $\sigma_F(\tilde{\mathbf{u}}, \widehat{q})\mathbf{n}$ is constant along Γ_{N}. Let $\mathbf{g} \in (\mathscr{C}_c^\infty(\Gamma_{\mathrm{N}}))^2$ fulfilling $\int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}\gamma_y = 0$. According to [72, Theorem IV.1.1], there exists $\mathbf{v}_{\mathbf{g}} \in \mathbf{H}^1(\mathscr{F}_0)$ satisfying

$$\begin{cases} \operatorname{div} \mathbf{v}_{\mathbf{g}} = 0 \text{ in } \mathscr{F}_0, \\ \mathbf{v}_{\mathbf{g}} = 0 \text{ on } \Gamma_{\mathrm{D}} \cup \partial S_0, \\ \mathbf{v}_{\mathbf{g}} = \mathbf{g} \text{ on } \Gamma_{\mathrm{N}}. \end{cases}$$

We know that $(\mathbf{v}_{\mathbf{g}}, 0, 0, 0, 0)$ belongs to $\mathbb{V}$. An integration by parts yields

$$\mathrm{a}_1(\mathbf{z}, (\mathbf{v}_{\mathbf{g}}, 0, 0, 0, 0)) = \int_{\mathscr{F}_0} (-\operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \widehat{q})) \cdot \mathbf{v}_{\mathbf{g}} \, \mathrm{d}\mathbf{y} + \int_{\Gamma_{\mathrm{N}}} \sigma_F(\tilde{\mathbf{u}}, \widehat{q})\mathbf{n} \cdot \mathbf{g} \, \mathrm{d}\gamma_y = \int_{\mathscr{F}_0} \mathbf{f}_{\mathbf{u}} \cdot \mathbf{v}_{\mathbf{g}} \, \mathrm{d}\mathbf{y},$$

and with (1.48) we get

$$\int_{\Gamma_{\mathrm{N}}} \sigma_F(\tilde{\mathbf{u}}, \widehat{q})\mathbf{n} \cdot \mathbf{g} \, \mathrm{d}\gamma_y = 0.$$

The previous equality holds for every $\mathbf{g} \in (\mathscr{C}_c^\infty(\Gamma_{\mathrm{N}}))^2$ fulfilling $\int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{n} \, \mathrm{d}\gamma_y = 0$, then there exists a constant c such that $\sigma_F(\tilde{\mathbf{u}}, \widehat{q})\mathbf{n} = c \, \mathbf{n}$ on Γ_{N}.

Let $q = \widehat{q} - c \in \mathrm{L}^2(\mathscr{F}_0)$, we have $\operatorname{div} \sigma_F(\tilde{\mathbf{u}}, q) = \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, \widehat{q})$ and $\sigma_F(\tilde{\mathbf{u}}, q)\mathbf{n} = 0$ on Γ_{N}. Moreover, $(\tilde{\mathbf{u}}, q)$ satisfies

$$\begin{cases} \operatorname{div} \sigma_F(\tilde{\mathbf{u}}, q) & \in \mathbf{L}^2(\mathscr{F}_0), \\ \operatorname{div} \tilde{\mathbf{u}} = 0 & \text{in } \mathscr{F}_0, \\ \tilde{\mathbf{u}} = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \tilde{\mathbf{u}} = \sum\limits_j \omega_j \partial_{\theta_j} \mathbf{\Phi^0}(0, 0, .) & \text{on } \partial S_0, \\ \sigma_F(\tilde{\mathbf{u}}, q)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{cases}$$

We finish this proof with a lifting of the boundary datum on ∂S_0 [118, Theorem 2.16] and Lemma 1.1.3. We get $D(A_1) \subset D(A)$, thus concluding the proof of Lemma 1.2.5. □

The key point of this section is the following lemma.

Lemma 1.2.6. *The operator A generates an analytic semigroup on $\mathbb{H}$. Moreover, for $\lambda \in \mathbb{R}$ large enough, $\lambda\mathrm{I} - A$ is positive and $D((\lambda\mathrm{I} - A)^{1/2}) = \mathbb{V}$.*

Proof. We first prove the properties of Lemma 1.2.6 on the self–adjoint operator A_1 and then we extend it to A with a perturbation argument.

The bilinear form a_1 is coercive and symmetric then $-A_1$ is non–negative and self–adjoint, so we can easily conclude that $D((-A_1)^{1/2}) = \mathbb{V}$.

Moreover, according to Korn's inequality [59, p. 110], there exists $c > 0$ such that

$$\forall(\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V},\ \mathrm{a}_1((\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2), (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)) + \frac{\nu}{2}\|\tilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\mathscr{F}_0)} \geq c\|\tilde{\mathbf{u}}\|^2_{\mathbf{H}^1(\mathscr{F}_0)},$$

so that,

$$\begin{aligned}&\forall(\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V},\\ &\mathrm{a}_1((\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2), (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)) + \max\left(\frac{\nu}{2}, c\right)\|(\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)\|^2_{\mathbb{H}} \geq c\|(\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)\|^2_{\mathbb{V}}.\end{aligned}$$

Hence, according to [24, Theorem 2.12, p. 115], A_1 generates an analytic semigroup on $\mathbb{H}$.

Now, we use the fact that $A - A_1 \in \mathcal{L}(\mathbb{H})$, then according to [125, Corollary 2.2.], A generates an analytic semigroup on $\mathbb{H}$.

A consequence of the previous result is that there exists $\lambda > 0$ such that $\lambda\mathrm{I} - A$ is positive. Moreover, $D(\lambda\mathrm{I} - A) = D(A_1)$, then by interpolation, $D((\lambda\mathrm{I} - A)^{1/2}) = D((-A_1)^{1/2}) = \mathbb{V}$. $\square$

We are now in position to prove Proposition 1.2.2.

Proof of Proposition 1.2.2. Let us denote $\mathbf{F} = \Pi_{\mathbb{H}}(\mathbf{f}, 0, 0, \mathcal{M}_{0,0}^{-1}\mathbf{s})$ and $\mathbf{z}_0 = (\mathbf{u}_0, 0, 0, \omega_{1,0}, \omega_{2,0})$. We have $\mathbf{F} \in \mathrm{L}^2(0, T; \mathbb{H})$ and $\mathbf{z}_0 \in D(A^{1/2}) = \mathbb{V}$.

According to [24, Theorem 3.1, p. 143] and Lemma 1.2.6, the problem

$$\left\{\begin{array}{ll}\mathbf{z}'(t) = A\mathbf{z}(t) + \mathbf{F}(t), & t \geq 0,\\ \mathbf{z}(0) = \mathbf{z}_0,\end{array}\right. \tag{1.49}$$

admits a unique solution $\mathbf{z} \in \mathrm{L}^2(0, T; D(A)) \cap \mathrm{H}^1(0, T; \mathbb{H})$ and there exists $C > 0$ such that

$$\|\mathbf{z}\|_{\mathrm{L}^2(0,T;D(A))\cap\mathrm{H}^1(0,T;\mathbb{H})} \leq C(\|\mathbf{F}\|_{\mathrm{L}^2(0,T;\mathbb{H})} + \|\mathbf{z}_0\|_{\mathbb{V}}). \tag{1.50}$$

With the Sobolev embedding

$$\mathrm{L}^2(0, T; D(A)) \cap \mathrm{H}^1(0, T; \mathbb{H}) \hookrightarrow \mathscr{C}^0([0, T]; \mathbb{V}),$$

we have

$$\|\mathbf{z}\|_{\mathrm{L}^2(0,T;D(A))\cap\mathscr{C}^0([0,T];\mathbb{V})\cap\mathrm{H}^1(0,T;\mathbb{H})} \leq C(\|\mathbf{F}\|_{\mathrm{L}^2(0,T;\mathbb{H})} + \|\mathbf{z}_0\|_{\mathbb{V}}). \tag{1.51}$$

Moreover, C is independent from $T \in (0, T_0)$. To prove this last statement, we consider

$$\forall t \in [0, T_0], \quad \tilde{\mathbf{F}}(t) = \left\{\begin{array}{ll}\mathbf{F}(t) & \text{if } t \in [0, T],\\ 0 & \text{if } t \in]T, T_0].\end{array}\right.$$

If $\tilde{\mathbf{z}}$ is the solution on $[0, T_0]$ of

$$\left\{\begin{array}{l}\tilde{\mathbf{z}}' = A\tilde{\mathbf{z}} + \tilde{\mathbf{F}},\\ \tilde{\mathbf{z}}(0) = \mathbf{z}_0,\end{array}\right.$$

then for $t \leq T$, $\tilde{\mathbf{z}}(t) = \mathbf{z}(t)$. And we have the inequality

$$\|\tilde{\mathbf{z}}\|_{\mathrm{L}^2(0,T_0;D(A))\cap\mathscr{C}^0([0,T_0];\mathbb{V})\cap\mathrm{H}^1(0,T_0;\mathbb{H})} \leq C(\|\tilde{\mathbf{F}}\|_{\mathrm{L}^2(0,T_0;\mathbb{H})} + \|\mathbf{z}_0\|_{\mathbb{V}}),$$

where C does not depend on T, while

$$\|\mathbf{z}\|_{\mathrm{L}^2(0,T;D(A))\cap\mathscr{C}^0([0,T];\mathbb{V})\cap\mathrm{H}^1(0,T;\mathbb{H})} \leq \|\tilde{\mathbf{z}}\|_{\mathrm{L}^2(0,T_0;D(A))\cap\mathscr{C}^0([0,T_0];\mathbb{V})\cap\mathrm{H}^1(0,T_0;\mathbb{H})},$$

and

$$\|\tilde{\mathbf{F}}\|_{\mathrm{L}^2(0,T_0;\mathbb{H})} = \|\mathbf{F}\|_{\mathrm{L}^2(0,T;\mathbb{H})}.$$

We get (1.51) with C independent from T.

Now, if we write $\mathbf{z} = (\tilde{\mathbf{u}}, \theta_1, \theta_2, \omega_1, \omega_2)$, problem (1.49) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{\mathbf{u}} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}}\begin{pmatrix} \mathrm{div}\ \sigma_F(\tilde{\mathbf{u}}, p) + \mathbf{f} \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1}\left(\mathbf{s} + \left(\int_{\partial S_0} -\sigma_F(\tilde{\mathbf{u}}, p)\mathbf{n}_0 \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{0}}(0,0,.)\ \mathrm{d}\gamma_y\right)_{j=1..2}\right) \end{pmatrix},$$

where $p \in \mathrm{L}^2(0,T;\mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0))$. Then, Lemma 1.2.3 implies that there exists $q \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathscr{F}_0))$ such that $(\tilde{\mathbf{u}}, p+q, \theta_1, \theta_2)$ satisfies the linear problem (1.36). Moreover, according to (1.51), we have $(\theta_1,\theta_2) \in \mathrm{H}^2(0,T;\mathbb{R}^2)$, $\tilde{\mathbf{u}} \in \mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0)) \cap \mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0)) \cap \mathrm{L}^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0))$, $\tilde{p} = p+q \in \mathrm{L}^2(0,T;\mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_0))$ and

$$\begin{aligned}\|\tilde{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_0))\cap\mathscr{C}^0([0,T];\mathbf{H}^1(\mathscr{F}_0))\cap\mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\tilde{p}\|_{\mathrm{L}^2(0,T;\mathbf{H}^{1/2+\varepsilon_0}(\mathscr{F}_0))} + \|(\theta_1,\theta_2)\|_{\Theta_T} \\ \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{s}\|_{\mathrm{L}^2(0,T;\mathbb{R}^2)}).\end{aligned}$$

We still have to show $\tilde{\mathbf{u}} \in \mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))$ and $\tilde{p} \in \mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0))$. According to [118, Theorem 2.16], there exists $\mathbf{v} \in \mathrm{H}^1(0,T;\mathbf{H}^2(\mathscr{F}_0))$ satisfying

$$\begin{cases} \mathrm{div}\ \mathbf{v} = 0 & \text{in } (0,T)\times\mathscr{F}_0, \\ \mathbf{v} = \sum_j \dot{\theta}_j \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{0}}(0,0,.) & \text{on } (0,T)\times\partial S_0, \\ \mathbf{v} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{D}}, \\ (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \end{cases}$$

with

$$\|\mathbf{v}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathscr{F}_0))} \leq C\|(\theta_1,\theta_2)\|_{\Theta_T}.$$

The velocity $\tilde{\mathbf{u}} - \mathbf{v}$ and the pressure $\tilde{p}$ satisfy for almost every t in $(0,T)$

$$\begin{cases} -\nu\Delta(\tilde{\mathbf{u}} - \mathbf{v}) + \nabla\tilde{p} = \mathbf{f} - \partial_t\tilde{\mathbf{u}} + \nu\Delta\mathbf{v} & \text{in } (0,T)\times\mathscr{F}_0, \\ \mathrm{div}\ (\tilde{\mathbf{u}} - \mathbf{v}) = 0 & \text{in } (0,T)\times\mathscr{F}_0, \\ \tilde{\mathbf{u}} - \mathbf{v} = 0 & \text{on } (0,T)\times(\Gamma_{\mathrm{D}}\cup\partial S_0), \\ \sigma_F(\tilde{\mathbf{u}} - \mathbf{v}, \tilde{p})\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \end{cases}$$

then, according to Lemma 1.1.3, $\tilde{\mathbf{u}} - \mathbf{v} \in \mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))$ and $\tilde{p} \in \mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0))$, where β is introduced in Lemma 1.1.3. Moreover, (1.30) yields

$$\|\tilde{\mathbf{u}} - \mathbf{v}\|_{\mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} + \|\tilde{p}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0))} \leq \|\mathbf{f} - \partial_t\tilde{\mathbf{u}} + \nu\Delta\mathbf{v}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0))}.$$

With the estimate $\|\tilde{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} \leq \|\tilde{\mathbf{u}} - \mathbf{v}\|_{\mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} + \|\mathbf{v}\|_{\mathrm{L}^2(0,T;\mathbf{H}^2(\mathscr{F}_0))}$, we get

$$\begin{aligned}\|\tilde{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{H}^2_\beta(\mathscr{F}_0))} + \|\tilde{p}\|_{\mathrm{L}^2(0,T;\mathrm{H}^1_\beta(\mathscr{F}_0))} \\ \leq C(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\tilde{\mathbf{u}}\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{v}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathscr{F}_0))}) \\ \leq C(\|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|\tilde{\mathbf{u}}\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_0))} + \|(\theta_1,\theta_2)\|_{\mathrm{H}^2(0,T;\mathbb{R}^2)}).\end{aligned}$$

This concludes the proof of Proposition 1.2.2. □

1.2.2 Linearized problem with nonhomogeneous boundary data

Let us now consider two more source terms : one source term $\mathbf{g}$ on the boundary of the structure ∂S_0 and one source term $\mathbf{u}^i$ on the inflow boundary region Γ_{i}. Let $T_0 > 0$, we study

$$\begin{cases} \dfrac{\partial \tilde{\mathbf{u}}}{\partial t} - \nu\Delta\tilde{\mathbf{u}} + \nabla\tilde{p} = \mathbf{f} & \text{in } (0,T)\times\mathscr{F}_0, \\ \operatorname{div}\tilde{\mathbf{u}} = 0 & \text{in } (0,T)\times\mathscr{F}_0, \\ \tilde{\mathbf{u}} = \dot{\theta}_1\partial_{\theta_1}\mathbf{\Phi^0}(0,0,.) + \dot{\theta}_2\partial_{\theta_2}\mathbf{\Phi^0}(0,0,.) + \mathbf{g} & \text{on } (0,T)\times\partial S_0, \\ \tilde{\mathbf{u}} = \mathbf{u}^i & \text{on } (0,T)\times\Gamma_{\mathrm{i}}, \\ \tilde{\mathbf{u}} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{w}}, \\ \sigma_F(\tilde{\mathbf{u}},\tilde{p})\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \\ \tilde{\mathbf{u}}(0,\mathbf{y}) = \mathbf{u}_0(\mathbf{y}) & \text{in } \mathscr{F}_0, \\ \mathcal{M}_{0,0}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \begin{pmatrix} \int_{\partial S_0}[\tilde{p}\mathrm{I} - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0\cdot\partial_{\theta_1}\mathbf{\Phi^0}(0,0,\gamma_y)\mathrm{d}\gamma_y \\ \int_{\partial S_0}[\tilde{p}\mathrm{I} - \nu(\nabla\tilde{\mathbf{u}} + (\nabla\tilde{\mathbf{u}})^T)]\mathbf{n}_0\cdot\partial_{\theta_2}\mathbf{\Phi^0}(0,0,\gamma_y)\mathrm{d}\gamma_y \end{pmatrix} + \mathbf{s} & \text{on } (0,T), \\ \theta_1(0) = 0, \quad \theta_2(0) = 0, \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}, \end{cases} \tag{1.52}$$

where the source terms are $\mathbf{f} \in \mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0))$, $\mathbf{g} \in \mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial S_0))$, $\mathbf{u}^i \in \mathrm{H}^1(0,T_0;\mathbf{U}^i)$ and $\mathbf{s} \in \mathrm{L}^2(0,T;\mathbb{R}^2)$.

We have the following result :

Proposition 1.2.7. *There exists a constant $C > 0$ such that for all $T \in (0,T_0)$, for all $\mathbf{u}^i \in \mathrm{H}^1(0,T_0;\mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0},\omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions* (1.35) *and every $(\mathbf{f},\mathbf{g},\mathbf{s}) \in \mathbb{F}_T \times \mathbb{G}_T \times \mathbb{S}_T$ with $\mathbf{g}(0) = 0$, problem* (1.52) *admits a unique solution*

$$(\tilde{\mathbf{u}},\tilde{p},\theta_1,\theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T,$$

with

$$\begin{aligned} \|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|(\theta_1,\theta_2)\|_{\Theta_T} \leq C(&\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| \\ &+\|\mathbf{f}\|_{\mathbb{F}_T} + \|\mathbf{g}\|_{\mathbb{G}_T} + \|\mathbf{s}\|_{\mathbb{S}_T} + \|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{U}^i)}). \end{aligned} \tag{1.53}$$

Proposition 1.2.7 is proven at the end of the section. The proof uses the following lifting result for the terms $\mathbf{g}$ and $\mathbf{u}^i$:

Lemma 1.2.8. *For every $\mathbf{g} \in \mathbf{H}^{3/2}(\partial S_0)$ and every $\mathbf{u}^i \in \mathbf{U}^i$, there exists $\overline{\mathbf{u}} \in \mathbf{H}^2(\mathscr{F}_0)$ satisfying*

$$\begin{cases} \operatorname{div}\overline{\mathbf{u}} = 0 & \text{in } \mathscr{F}_0, \\ \overline{\mathbf{u}} = \mathbf{g} & \text{on } \partial S_0, \\ \overline{\mathbf{u}} = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\ \overline{\mathbf{u}} = 0 & \text{on } \Gamma_{\mathrm{w}}, \\ (\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{1.54}$$

with

$$\|\overline{\mathbf{u}}\|_{\mathbf{H}^2(\mathscr{F}_0)} \leq C(\|\mathbf{u}^i\|_{\mathbf{U}^i} + \|\mathbf{g}\|_{\mathbf{H}^{3/2}(\partial S_0)}). \tag{1.55}$$

Note that despite the presence of corners, we recover the expected regularity of the lifting for smooth domains.

Remark 1.2.9. For the sake of readability, from this point onwards all terms $\mathrm{d}\mathbf{y}$ and $\mathrm{d}\gamma_y$ are omitted in the integrals.

Proof of Lemma 1.2.8. The lifting result has been established for the condition $\mathbf{u}^i = 0$ on the inflow region in [118, Theorem 2.16]. We first lift the inflow boundary condition $\mathbf{u}^i \neq 0$ in Ω and then we use the aforementioned result.

Lifting of the inflow boundary condition. Let us look for a function $\mathbf{v}$ defined on the entire domain Ω and satisfying

$$\begin{cases} \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\ \mathbf{v} = 0 & \text{on } \Gamma_{\mathrm{w}}, \\ (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{cases} \tag{1.56}$$

As $\mathbf{v}$ is divergence–free and Ω is simply connected, we look for it under the form $\mathbf{v} = \nabla^{\perp}\psi$, where ψ is a scalar–valued function. In the geometry considered, Γ_{i}, Γ_{t}, Γ_{b} and Γ_{N} are straight lines, hence $\partial_{\mathbf{n}_0}$ is written as $\pm\partial_{y_1}$ or $\pm\partial_{y_2}$ according on the considered part of the boundary.

We can prove that ψ has to satisfy the conditions

$$\begin{array}{lll} \partial_{y_2}\psi = -u_1^i & \text{and } \partial_{y_1}\psi = u_2^i & \text{on } \Gamma_{\mathrm{i}}, \\ \partial_{y_1}\psi = 0 & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{b}}, \\ \partial_{y_1}\psi = 0 & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{t}}, \\ \partial_{y_1}\partial_{y_2}\psi = 0 & \text{and } \partial^2_{y_1}\psi - \partial^2_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{array}$$

We choose to meet these conditions in the following way :

$$\begin{array}{llll} & \psi(y_2) = -\displaystyle\int_0^{y_2} u_1^i & \text{and } \partial_{y_1}\psi = u_2^i & \text{on } \Gamma_{\mathrm{i}}, \\ & \psi = 0 & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{b}}, \\ & \psi = -\displaystyle\int_{\Gamma_{\mathrm{i}}} u_1^i & \text{and } \partial_{y_2}\psi = 0 & \text{on } \Gamma_{\mathrm{t}}. \\ \psi(y_2) = -\eta(y_2)\displaystyle\int_{\Gamma_{\mathrm{i}}} u_1^i, & \partial_{y_1}\psi = 0 & \text{and } \partial^2_{y_1}\psi = -\mathrm{d}^2_{y_2}\eta(y_2)\displaystyle\int_{\Gamma_{\mathrm{i}}} u_1^i & \text{on } \Gamma_{\mathrm{N}}, \end{array} \tag{1.57}$$

where η is a $\mathscr{C}^\infty$ function on $[0,1]$ satisfying

$$\forall\, y_2 \in [0,1], \quad \eta(y_2) = \begin{cases} 0 & \text{if } \ y_2 \in [0, 1/4], \\ \in [0,1] & \text{if } \ y_2 \in]1/4, 3/4[, \\ 1 & \text{if } \ y_2 \in [3/4, 1]. \end{cases} \tag{1.58}$$

The theorem [80, Theorem 1.6.1.5, p.69] with $m = 3$ and $d = 2$ gives the existence of $\psi \in \mathrm{H}^3(\Omega)$ fulfilling (1.57) under the compatibility conditions :

$$\text{there exist } \alpha_1 \text{ and } \alpha_2 > 0 \text{ such that } \begin{cases} \displaystyle\int_0^{\alpha_1} \frac{|\partial_{y_2} u_2^i|^2}{y_2} < +\infty, \\ \displaystyle\int_{1-\alpha_2}^1 \frac{|\partial_{y_2} u_2^i|^2}{1 - y_2} < +\infty. \end{cases} \tag{1.59}$$

These conditions are the ones in the definition of $\mathbf{U}^i$ in (1.34) with $\alpha_1 = \alpha_2 = 1/4$. Moreover we have the estimate

$$\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)} \leq c\|\psi\|_{\mathrm{H}^3(\Omega)} \leq C\|\mathbf{u}^i\|_{\mathbf{H}^{3/2}(\Gamma_{\mathrm{i}})}. \tag{1.60}$$

The divergence-free field $\mathbf{v} = \nabla^\perp \psi \in \mathbf{H}^2(\Omega)$ satisfies (1.56).

Lifting of the structure velocity. Now, $\tilde{\mathbf{v}} = \overline{\mathbf{u}} - \mathbf{v}_{|\mathscr{F}_0}$ has to satisfy

$$\begin{cases} \operatorname{div} \tilde{\mathbf{v}} = 0 & \text{in } \mathscr{F}_0, \\ \tilde{\mathbf{v}} = \mathbf{g} - \mathbf{v} & \text{on } \partial S_0, \\ \tilde{\mathbf{v}} = 0 & \text{on } \Gamma_{\mathrm{i}}, \\ \tilde{\mathbf{v}} = 0 & \text{on } \Gamma_{\mathrm{w}}, \\ (\nabla \tilde{\mathbf{v}} + (\nabla \tilde{\mathbf{v}})^T)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{cases}$$

According to [118, Theorem 2.16], such $\tilde{\mathbf{v}}$ exists in $\mathbf{H}^2(\mathscr{F}_0)$ as soon as $\mathbf{g} - \mathbf{v} \in \mathbf{H}^{3/2}(\partial S_0)$. Moreover, we have the estimate

$$\|\tilde{\mathbf{v}}\|_{\mathbf{H}^2(\mathscr{F}_0)} \leq C\|\mathbf{g} - \mathbf{v}\|_{\mathbf{H}^{3/2}(\partial S_0)} \leq C(\|\mathbf{g}\|_{\mathbf{H}^{3/2}(\partial S_0)} + \|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}). \tag{1.61}$$

This yields the expected result since $\overline{\mathbf{u}} = \tilde{\mathbf{v}} + \mathbf{v}_{|\mathscr{F}_0}$, the estimate (1.55) comes from (1.60) and (1.61). □

We can now prove Proposition 1.2.7 in the following way.

Proof of Proposition 1.2.7. Let $\mathbf{u}^i \in \mathrm{H}^1(0, T_0; \mathbf{U}^i)$, $\mathbf{u}_0 \in \mathbf{H}^1(\mathscr{F}_0)$ and $(\omega_{1,0}, \omega_{2,0}) \in \mathbb{R}^2$ satisfying the compatibility conditions (1.35). Let $(\mathbf{f}, \mathbf{g}, \mathbf{s}) \in \mathbb{F}_T \times \mathbb{G}_T \times \mathbb{S}_T$ with $\mathbf{g}(0) = 0$.

Let $\overline{\mathbf{u}} \in \mathrm{H}^1(0, T; \mathbf{H}^2(\mathscr{F}_0))$ be the solution to (1.54), it fulfils

$$\|\overline{\mathbf{u}}\|_{\mathrm{H}^1(0,T;\mathbf{H}^2(\mathscr{F}_0))} \leq C(\|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{g}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial S_0))}). \tag{1.62}$$

The lifting $\overline{\mathbf{u}}$ also belongs to $\mathscr{C}^0([0, T]; \mathbf{H}^2(\mathscr{F}_0))$, and as $\mathbf{g}(0) = 0$, we have

$$\|\overline{\mathbf{u}}\|_{\mathscr{C}^0([0,T];\mathbf{H}^2(\mathscr{F}_0))} \leq C(\|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{g}\|_{\mathrm{H}^1(0,T;\mathbf{H}^{3/2}(\partial S_0))}), \tag{1.63}$$

where C does not depend on T.

Let $(\hat{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ be the solution to

$$\begin{cases} \dfrac{\partial \hat{\mathbf{u}}}{\partial t} - \nu \Delta \hat{\mathbf{u}} + \nabla \tilde{p} = \mathbf{f} - \dfrac{\partial \overline{\mathbf{u}}}{\partial t} + \nu \Delta \overline{\mathbf{u}} & \text{in } (0,T) \times \mathscr{F}_0, \\ \operatorname{div} \hat{\mathbf{u}} = 0 & \text{in } (0,T) \times \mathscr{F}_0, \\ \hat{\mathbf{u}} = \dot{\theta}_1 \partial_{\theta_1} \mathbf{\Phi^0}(0,0,.) + \dot{\theta}_2 \partial_{\theta_2} \mathbf{\Phi^0}(0,0,.) & \text{on } (0,T) \times \partial S_0, \\ \hat{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{i}}, \\ \hat{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{w}}, \\ \sigma_F(\hat{\mathbf{u}}, \tilde{p})\mathbf{n} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{N}}, \\ \hat{\mathbf{u}}(0,.) = \mathbf{u}_0(.) - \overline{\mathbf{u}}(0,.) & \text{in } \mathscr{F}_0, \\ \mathcal{M}_{0,0} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}) + (\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}))^T)]\mathbf{n}_0 \cdot \partial_{\theta_1} \mathbf{\Phi^0}(0,0,\gamma_y) \\ \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}) + (\nabla(\hat{\mathbf{u}} + \overline{\mathbf{u}}))^T)]\mathbf{n}_0 \cdot \partial_{\theta_2} \mathbf{\Phi^0}(0,0,\gamma_y) \end{pmatrix} + \mathbf{s} & \text{on } (0,T), \\ \theta_1(0) = 0, \quad \theta_2(0) = 0, \\ \dot{\theta}_1(0) = \omega_{1,0}, \quad \dot{\theta}_2(0) = \omega_{2,0}. \end{cases}$$

We have

$$\begin{aligned} &\mathbf{f} - \frac{\partial \overline{\mathbf{u}}}{\partial t} + \nu \Delta \overline{\mathbf{u}} \in \mathrm{L}^2(0, T; \mathbf{L}^2(\mathscr{F}_0)), \\ &\mathbf{u}_0(.) - \overline{\mathbf{u}}(0,.) = 0 \text{ on } \Gamma_{\mathrm{i}}, \\ &s_j + \int_{\partial S_0} -\nu(\nabla \hat{\mathbf{u}} + (\nabla \hat{\mathbf{u}})^T)\mathbf{n}_0 \cdot \partial_{\theta_j} \mathbf{\Phi^0}(0,0,\gamma_y) \in \mathrm{L}^2(0,T). \end{aligned}$$

Then, according to Proposition 1.2.2, $(\hat{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ and we have (1.43) with $\hat{\mathbf{u}} = \tilde{\mathbf{u}}$.

Now, we consider $\tilde{\mathbf{u}} = \hat{\mathbf{u}} + \overline{\mathbf{u}}$, then $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ and (1.53) is a consequence of (1.62)–(1.63) and (1.43). □

Note that a larger space than $\mathrm{H}^1(0, T_0; \mathbf{U}^i)$ could be considered for $\mathbf{u}^i$. Indeed, we use a lifting in space only, inducing the requirement $\mathbf{u}^i \in \mathrm{H}^1(0, T_0; \mathbf{U}^i)$. Using a space-time lifting would be slightly more complicated (see [130]), but would allow a larger space for the inflow boundary datum $\mathbf{u}^i$.

1.3 Local existence of solution to the full problem

In this section, we study the nonlinear problem. We recall that $\theta_{1,0} = \theta_{2,0} = 0$. At first, we rewrite the equations (1.25) in the fixed domain $\mathscr{F}_0$, then we prove existence of a solution to this problem.

1.3.1 The equations in a fixed domain

Our goal is to write the equations (1.25) in the fixed domain $\mathscr{F}_0$. To do so, we use the diffeomorphism defined in (1.31). We denote $\mathcal{J}_{\mathbf{\Phi^0}}$ its Jacobian matrix and $\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}})$ the cofactor matrix of $\mathcal{J}_{\mathbf{\Phi^0}}$. We use the change of variables

$$\forall t \in [0, T], \quad \forall \mathbf{y} \in \mathscr{F}_0, \quad \left\{ \begin{array}{l} \tilde{\mathbf{u}}(t, \mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1(t), \theta_2(t), \mathbf{y}))^T \mathbf{u}(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y})), \\ \tilde{p}(t, \mathbf{y}) = p(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y})). \end{array} \right.$$

This choice is motivated by the fact that, according to [34, Lemma 3.1], we get $\mathrm{div}\, \tilde{\mathbf{u}} = 0$.

In the sequel, v_i denotes the i^{th} component of the vector $\mathbf{v}$. We recall that $\mathbf{\Psi^0}(\theta_1, \theta_2, .)$ is the inverse diffeomorphism of $\mathbf{\Phi^0}(\theta_1, \theta_2, .)$. To compute the equations satisfied by $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$, we use the following explicit formula :

$$\mathbf{u}(t, \mathbf{x}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})),$$

we have

$$\begin{aligned} \partial_t \mathbf{u}(t, \mathbf{x}) = {} & \mathrm{cof}\left(\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x})\right)^T \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \partial_t \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}), \end{aligned}$$

$$\begin{aligned} \partial_{x_j} \mathbf{u}(t, \mathbf{x}) = {} & \mathrm{cof}(\partial_{x_j} \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \partial_{x_j} \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}), \end{aligned}$$

and

$$\begin{aligned} \partial^2_{x_j} \mathbf{u}(t, \mathbf{x}) = {} & \mathrm{cof}(\partial^2_{x_j} \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \\ & + 2\mathrm{cof}(\partial_{x_j} \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \partial_{x_j} \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \sum_k \partial_{y_k} \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \partial_{x_j} \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}) \\ & \qquad \partial_{x_j} \Psi^0_k(\theta_1(t), \theta_2(t), \mathbf{x}) \\ & + \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x})) \partial^2_{x_j} \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}). \end{aligned}$$

Problem (1.25) in the fixed domain reads (1.52) where $\mathbf{f}, \mathbf{g}, \mathbf{s}$ are defined by

$$\left\{ \begin{array}{l} \mathbf{f} = \mathbf{F}(\theta_1, \theta_2, \tilde{\mathbf{u}}, \tilde{p}) + \mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\theta_1(t), \theta_2(t), \mathbf{y})), \\ \mathbf{g} = \mathbf{G}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2), \\ \mathbf{s} = \mathbf{S}(\theta_1, \theta_2, \tilde{\mathbf{u}}, \tilde{p}) + \mathbf{f_s}, \end{array} \right. \tag{1.64}$$

and we can decompose $\mathbf{F}(\theta_1, \theta_2, \tilde{\mathbf{u}}, \tilde{p}) = \mathbf{F}^1 + \mathbf{F}^2 + \mathbf{F}^3 + \mathbf{F}^4 + \mathbf{F}^5$, where $\mathbf{F}^i$, $\mathbf{G}$ and $\mathbf{S}$ are given below in (1.65)–(1.66).

We write $\mathbf{\Phi^0}(\theta_1, \theta_2, .)$ under the simpler notation $\mathbf{\Phi^0}$. The nonlinear terms are given as follows :

$$\begin{aligned}
\mathbf{F}^1(\theta_1, \theta_2, \tilde{\mathbf{u}}) &= (\mathrm{I} - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))^T) \frac{\partial \tilde{\mathbf{u}}}{\partial t}, \\
\mathbf{F}^2(\theta_1, \theta_2, \tilde{\mathbf{u}}) &= -\mathrm{cof}(\dot{\theta}_1 \nabla_{\mathbf{x}} \partial_{\theta_1} \mathbf{\Psi^0}(\theta_1, \theta_2, \mathbf{\Phi^S}) + \dot{\theta}_2 \nabla_{\mathbf{x}} \partial_{\theta_2} \mathbf{\Psi^0}(\theta_1, \theta_2, \mathbf{\Phi^S}))^T \tilde{\mathbf{u}}(t, \mathbf{y}) \\
&\quad -\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))^T (\nabla_{\mathbf{y}} \tilde{\mathbf{u}}) \Big(\dot{\theta}_1 \partial_{\theta_1} \mathbf{\Psi^0}(\theta_1, \theta_2, \mathbf{\Phi^0}) + \dot{\theta}_2 \partial_{\theta_2} \mathbf{\Psi^0}(\theta_1, \theta_2, \mathbf{\Phi^0}) \Big), \\
\mathbf{F}^3(\theta_1, \theta_2, \tilde{\mathbf{u}})_i &= \nu \sum_{j,k,l,m} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))_{ki} \frac{\partial^2 \tilde{u}_k}{\partial y_l \partial y_m} \frac{\partial \Psi^0_l}{\partial x_j}(\theta_1, \theta_2, \mathbf{\Phi^0}) \frac{\partial \Psi^0_m}{\partial x_j}(\theta_1, \theta_2, \mathbf{\Phi^0}) \\
&\quad + 2\nu \sum_{j,k,l} \frac{\partial}{\partial x_j} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))_{ki} \frac{\partial \tilde{u}_k}{\partial y_l} \frac{\partial \Psi^0_l}{\partial x_j}(\theta_1, \theta_2, \mathbf{\Phi^0}) \\
&\quad + \nu \sum_{j,k,l} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))_{ki} \frac{\partial \tilde{u}_k}{\partial y_l} \frac{\partial^2 \Psi^0_l}{\partial x_j^2}(\theta_1, \theta_2, \mathbf{\Phi^0}) \\
&\quad + \nu \sum_{j,k} \frac{\partial^2}{\partial x_j^2} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))_{ki} \tilde{u}_k - \nu \Delta_{\mathbf{y}} \tilde{u}_i(t, \mathbf{y}),
\end{aligned} \tag{1.65}$$

$$\begin{aligned}
\mathbf{F}^4(\theta_1, \theta_2, \tilde{\mathbf{u}})_i &= -\sum_{j,k,r} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))_{kj} \frac{\partial}{\partial x_j} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2))_{ri} \tilde{u}_k \tilde{u}_r \\
&\quad - \sum_{k,r} \det(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))^2 \frac{\partial \Phi^0_i}{\partial y_r} \tilde{u}_k \frac{\partial \tilde{u}_r}{\partial y_k}, \\
\mathbf{F}^5(\theta_1, \theta_2, \tilde{p}) &= (\mathrm{I} - \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))^T \nabla_{\mathbf{y}} \tilde{p}, \\
\mathbf{G}(\theta_1, \theta_2, \omega_1, \omega_2) &= \sum_{j=1}^{2} \omega_j \Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1, \theta_2))^T \partial_{\theta_j} \mathbf{\Phi^0}(\theta_1, \theta_2, \mathbf{y}) - \partial_{\theta_j} \mathbf{\Phi^0}(0, 0, \mathbf{y}) \Big), \\
\mathbf{S}(\theta_1, \theta_2, \tilde{\mathbf{u}}, \tilde{p}) &= -(\mathcal{M}_{\theta_1, \theta_2} - \mathcal{M}_{0,0}) \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \mathbf{M_I}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\
&\quad + \begin{pmatrix} \displaystyle\int_{\partial S_0} |\mathcal{J}_{\mathbf{\Phi^0}} \mathbf{t}_0| \, [\tilde{p}\mathrm{I} - \nu(\mathcal{G}(\theta_1, \theta_2, \tilde{\mathbf{u}}) + \mathcal{G}(\theta_1, \theta_2, \tilde{\mathbf{u}})^T)] \mathbf{n}_{\theta_1, \theta_2}(\mathbf{\Phi^0}) \cdot \partial_{\theta_1} \mathbf{\Phi^0}(\theta_1, \theta_2, \gamma_y) \\ \displaystyle\int_{\partial S_0} |\mathcal{J}_{\mathbf{\Phi^0}} \mathbf{t}_0| \, [\tilde{p}\mathrm{I} - \nu(\mathcal{G}(\theta_1, \theta_2, \tilde{\mathbf{u}}) + \mathcal{G}(\theta_1, \theta_2, \tilde{\mathbf{u}})^T)] \mathbf{n}_{\theta_1, \theta_2}(\mathbf{\Phi^0}) \cdot \partial_{\theta_2} \mathbf{\Phi^0}(\theta_1, \theta_2, \gamma_y) \end{pmatrix} \\
&\quad - \begin{pmatrix} \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T)] \mathbf{n}_0 \cdot \partial_{\theta_1} \mathbf{\Phi^0}(0, 0, \gamma_y) \\ \displaystyle\int_{\partial S_0} [\tilde{p}\mathrm{I} - \nu(\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T)] \mathbf{n}_0 \cdot \partial_{\theta_2} \mathbf{\Phi^0}(0, 0, \gamma_y) \end{pmatrix},
\end{aligned} \tag{1.66}$$

where $\mathbf{t}_0$ is a unitary tangent vector to ∂S_0, $\mathbf{M_I}$ and $\mathcal{M}_{\theta_1, \theta_2}$ are defined in (1.17), (1.18) and

$$\begin{aligned}
\mathcal{G}(\theta_1, \theta_2, \tilde{\mathbf{u}})_{ij} &= \sum_k \mathrm{cof} \left[\partial_{x_j} \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, .)) \circ \mathbf{\Phi^0} \right]_{ki} \tilde{u}_k \\
&\quad + \sum_{k,l} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1, \theta_2, \mathbf{\Phi^0}))_{ki} \frac{\partial \tilde{u}_k}{\partial y_l} \frac{\partial \Psi^0_l}{\partial x_j}(\theta_1, \theta_2, \mathbf{\Phi^0}).
\end{aligned} \tag{1.67}$$

We can state the following theorem.

Theorem 1.3.1. *Let $T_0 > 0$. Let $\mathbf{u}^i \in \mathrm{H}^1(0, T_0; \mathbf{U}^i)$. For every $(\mathbf{u}_0, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_0) \times \mathbb{R}^2$ satisfying the compatibility conditions* (1.35)*, there exists $T \in (0, T_0]$ such that for every $(\mathbf{f}_{\mathscr{F}}, \mathbf{f_s}) \in \mathrm{L}^2(0, T; \mathbf{W}^{1,\infty}(\Omega)) \times \mathrm{L}^2(0, T; \mathbb{R}^2)$ problem* (1.52) *where the source terms are given by* (1.64) *admits a unique solution $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times (\Theta_T \cap \mathrm{L}^2(0, T; \mathbb{D}_\Theta))$ satisfying the following estimate*

$$\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|(\theta_1, \theta_2)\|_{\Theta_T} \leq C(\|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_{\mathscr{F}}\|_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{f_s}\|_{\mathrm{L}^2(0,T_0)}),$$

where C does not depend on T, $\mathbf{f}_{\mathscr{F}}$, $\mathbf{f_s}$ and $\mathbf{u}^i$.

This theorem is the rewriting of Theorem 1.1.5 in the fixed domain $\mathscr{F}_0$. To prove Theorem 1.3.1, we use the results of Section 1.2 and a fixed point argument.

1.3.2 Proof of Theorem 1.3.1

Proof. We work in the fixed fluid domain $\mathscr{F}_0$. Let $T_0 > 0$.
Let $\mathbf{u}^i \in \mathrm{H}^1(0, T_0; \mathbf{U}^i)$ and $(\mathbf{u}_0, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_0) \times \mathbb{R}^2$ satisfying the compatibility conditions (1.35).

We define the space

$$\mathbb{N}_T = \{(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T \ \text{ with } \ \theta_1(0) = \theta_2(0) = 0, \quad \forall t \in [0, T], \ \ (\theta_1, \theta_2)(t) \in \mathbb{D}_\Theta\},$$

endowed with the norm

$$\|(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)\|_{\mathbb{N}_T} = \|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|(\theta_1, \theta_2)\|_{\Theta_T}. \tag{1.68}$$

We also define an application Λ^T on $\mathbb{N}_T$ such that for every $(\overline{\mathbf{u}}, \overline{p}, \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{N}_T$, $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) = \Lambda^T(\overline{\mathbf{u}}, \overline{p}, \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ is the solution to problem (1.52), where the nonhomogeneous terms are given by

$$\begin{aligned}
\mathbf{f} &= \mathbf{F}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{u}}, \overline{p}) + \mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\overline{\theta}_1, \overline{\theta}_2, \mathbf{y})),\\
\mathbf{g} &= \mathbf{G}(\overline{\theta}_1, \overline{\theta}_2, \dot{\overline{\theta}}_1, \dot{\overline{\theta}}_2),\\
\mathbf{s} &= \mathbf{S}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{u}}, \overline{p}) + \mathbf{f_s},
\end{aligned}$$

where $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{S}$ are given by (1.66). If $(., ., \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{N}_T$, then $\mathbf{G}(\overline{\theta}_1, \overline{\theta}_2, \dot{\overline{\theta}}_1, \dot{\overline{\theta}}_2)(t = 0) = 0$, then according to Proposition 1.2.7, the application Λ^T is well defined. Note that Λ^T depends on the initial data $(\mathbf{u}_0, \omega_{1,0}, \omega_{2,0})$ and on the source term $\mathbf{u}^i$.

We take

$$R = 2C(\|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}_{\mathscr{F}}\|_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{f_s}\|_{\mathrm{L}^2(0,T_0)}),$$

where C is the constant of Proposition 1.2.7, so that Proposition 1.2.7 gives

$$\begin{aligned}\|\Lambda^T(0, 0, 0, 0)\|_{\mathbb{N}_T} \leq C(&\|\mathbf{u}^i\|_{\mathrm{H}^1(0,T_0;\mathbf{U}^i)} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\mathscr{F}_0)} + |\omega_{1,0}| + |\omega_{2,0}| \\ &+ \|\mathbf{f}_{\mathscr{F}}\|_{\mathrm{L}^2(0,T_0;\mathbf{L}^2(\mathscr{F}_0))} + \|\mathbf{f_s}\|_{\mathrm{L}^2(0,T_0)}) = R/2.\end{aligned} \tag{1.69}$$

The strategy adopted is based on the existence of $T > 0$ such that Λ^T is a contraction on

$$\mathbb{B}_R(T) = \left\{ \ (\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2) \in \mathbb{N}_T \quad \text{ with } \quad \|(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)\|_{\mathbb{N}_T} \leq R \ \right\}. \tag{1.70}$$

Remark 1.3.2. The domain $\mathbb{D}_\Theta$ is an open subset of $\mathbb{R}^2$ and $(0,0) \in \mathbb{D}_\Theta$, then there exists $r > 0$ such that $B((0,0),r) \subset \mathbb{D}_\Theta$. Then for $T < r/R$, if $\|\dot{\theta}_j\|_{\mathrm{L}^\infty(0,T)} \le R$ and $\theta_j(0) = 0$, we have

$$\|\theta_j\|_{\mathrm{L}^\infty(0,T)} \le T\|\dot{\theta}_j\|_{\mathrm{L}^\infty(0,T)} \le RT \le r,$$

and we have for all $t \in (0,T)$, $(\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta$. In the sequel we choose $T_0 > 0$ such that $T_0 < r/R$.

The solution to the nonlinear problem will be obtained as a fixed point of the application Λ^T. We use the estimates of the following lemma.

Lemma 1.3.3. *For every $R' > 0$, there exists a constant $C' = C'(R') > 0$, such that for every $T \in (0,T_0)$, and every $(\tilde{\mathbf{u}}^j, \tilde{p}^j, \theta_1^j, \theta_2^j) \in \widetilde{\mathbb{B}_R^\infty}$, we have*

$$\|\mathbf{F}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{F}_T} \le C'T^{1/4}(\|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T} + \|\tilde{p}^a - \tilde{p}^b\|_{\mathbb{P}_T} + \|\theta^a - \theta^b\|_{\Theta_T}), \tag{1.71}$$
$$\|\mathbf{G}(\theta_1^a, \theta_2^a, \dot{\theta}_1^a, \dot{\theta}_2^a) - \mathbf{G}(\theta_1^b, \theta_2^b, \dot{\theta}_1^b, \dot{\theta}_2^b)\|_{\mathbb{G}_T} \le C'T\|\theta^a - \theta^b\|_{\Theta_T}, \tag{1.72}$$
$$\|\mathbf{S}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{S}_T} \le C'T^{1/2}(\|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T} + \|\tilde{p}^a - \tilde{p}^b\|_{\mathbb{P}_T} + \|\theta^a - \theta^b\|_{\Theta_T}), \tag{1.73}$$
$$\|\mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\theta_1^a, \theta_2^a, \mathbf{y})) - \mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\theta_1^b, \theta_2^b, \mathbf{y}))\|_{\mathbb{F}_T} \le C'T\|\theta^a - \theta^b\|_{\Theta_T}. \tag{1.74}$$

These estimates are proven in Appendix A.

For $(\tilde{\mathbf{u}}^j, \tilde{p}^j, \theta_1^j, \theta_2^j) \in \mathbb{B}_R(T)$, Proposition 1.2.7 yields the estimate

$$\begin{aligned}
&\|\Lambda^T(\tilde{\mathbf{u}}^a, \tilde{p}^a, \theta_1^a, \theta_2^a) - \Lambda^T(\tilde{\mathbf{u}}^b, \tilde{p}^b, \theta_1^b, \theta_2^b)\|_{\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T} \\
&\quad \le C(\|\mathbf{F}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{F}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{F}_T} + \|\mathbf{G}(\theta_1^a, \theta_2^a, \dot{\theta}_1^a, \dot{\theta}_2^a) - \mathbf{G}(\theta_1^b, \theta_2^b, \dot{\theta}_1^b, \dot{\theta}_2^b)\|_{\mathbb{G}_T} \\
&\quad + \|\mathbf{S}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a, \tilde{p}^a) - \mathbf{S}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b, \tilde{p}^b)\|_{\mathbb{S}_T} + \|\mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\theta_1^a, \theta_2^a, \mathbf{y})) - \mathbf{f}_{\mathscr{F}}(t, \mathbf{\Phi^0}(\theta_1^b, \theta_2^b, \mathbf{y}))\|_{\mathbb{F}_T}),
\end{aligned} \tag{1.75}$$

and with Lemma 1.3.3, we have

$$\begin{aligned}
\|\Lambda^T(\tilde{\mathbf{u}}^a, \tilde{p}^a, \theta_1^a, \theta_2^a) - \Lambda^T(\tilde{\mathbf{u}}^b, \tilde{p}^b, \theta_1^b, \theta_2^b)\|_{\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T} \le KT^{1/4}(\|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T} + \\
\|\tilde{p}^a - \tilde{p}^b\|_{\mathbb{P}_T} + \|\theta^a - \theta^b\|_{\Theta_T}),
\end{aligned} \tag{1.76}$$

where $K = 4CC'(R)$ depends on R but not on T. Then, for $T \in (0,T_0)$ such that

$$KT^{1/4} \le 1/2,$$

the application Λ^T is a contraction on $\mathbb{B}_R(T)$. Moreover, (1.76) and (1.69) yield

$$\begin{aligned}
\|\Lambda^T(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)\|_{\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T} &\le \|\Lambda^T(0,0,0,0)\|_{\mathbb{N}_T} + KT^{1/4}(\|\tilde{\mathbf{u}}\|_{\mathbb{U}_T} + \|\tilde{p}\|_{\mathbb{P}_T} + \|\theta\|_{\Theta_T}) \\
&\le R/2 + KRT^{1/4} \le R.
\end{aligned}$$

According to Remark 1.3.2, we have proven that $\Lambda^T : \mathbb{B}_R(T) \to \mathbb{B}_R(T)$ is a contraction. Then, according to the Picard fixed point theorem, there exists a unique fixed point to Λ^T in $\mathbb{B}_R(T)$. This fixed point is the solution to problem (1.52) where the source terms are given by (1.64). This proves Theorem 1.3.1. $\square$

1.3.3 Proof of the result in the moving domain, Theorem 1.1.5

We consider $(\tilde{\mathbf{u}}, \tilde{p}, \theta_1, \theta_2)$ in $\mathbb{U}_T \times \mathbb{P}_T \times \Theta_T$ the solution to problem (1.52) with (1.64) given by Theorem 1.3.1. Let $\mathbf{u}(t,\mathbf{x}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \tilde{\mathbf{u}}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}))$ and $p(t,\mathbf{x}) = \tilde{p}(t, \mathbf{\Psi^0}(\theta_1(t), \theta_2(t), \mathbf{x}))$. Then the quadruplet $(\mathbf{u}, p, \theta_1, \theta_2)$ is solution to the problem in the moving domain. This proves Theorem 1.1.5.

Chapitre 2

Stabilisation locale du système d'interaction fluide–structure autour d'un état stationnaire

Abstract. We study the stabilization of solutions to a 2d fluid–structure system by a feedback control law acting on the acceleration of the structure. The structure is described by a finite number of parameters. The modelling of this system and the existence of strong solutions has been previously studied in Chapter 1. We consider an unstable stationary solution to the problem. We assume a unique continuation property for the eigenvectors of the adjoint system. Under this assumption, the nonlinear feedback control that we propose stabilizes the whole fluid–structure system around the stationary solution at any chosen exponential decay rate for small enough initial perturbations. Our method reposes on the analysis of the linearized system and the feedback operator is given by a Riccati equation of small dimension.

MSC numbers. 35Q30, 74F10, 76D55, 93D15

2.1 Introduction

The goal of this study is to stabilize a 2d fluid–structure interaction problem. The fluid is modelled by the incompressible Navier–Stokes equations and the structure, immersed in the fluid, is governed by a finite number of parameters. Such a kind of structure can be found for instance in aeronautics [95]. Our goal is to design a finite dimensional feedback controller which stabilizes locally the system around a given stationary state at any prescribed exponential decay rate.

In order to simplify the study, we consider that the structure is described by only two parameters θ_1 and θ_2. However, all the results that we present in the sequel can be easily extended to the case of a structure depending on N (≥ 1) parameters (see Remark 2.1.1).

2.1.1 Modelling of the problem

The fluid–structure configuration considered in this paper has already been investigated in Chapter 1 where existence of strong solutions has been proven. We consider a bounded domain $\Omega = (0, L) \times (0, 1)$ (see Fig.2.1).

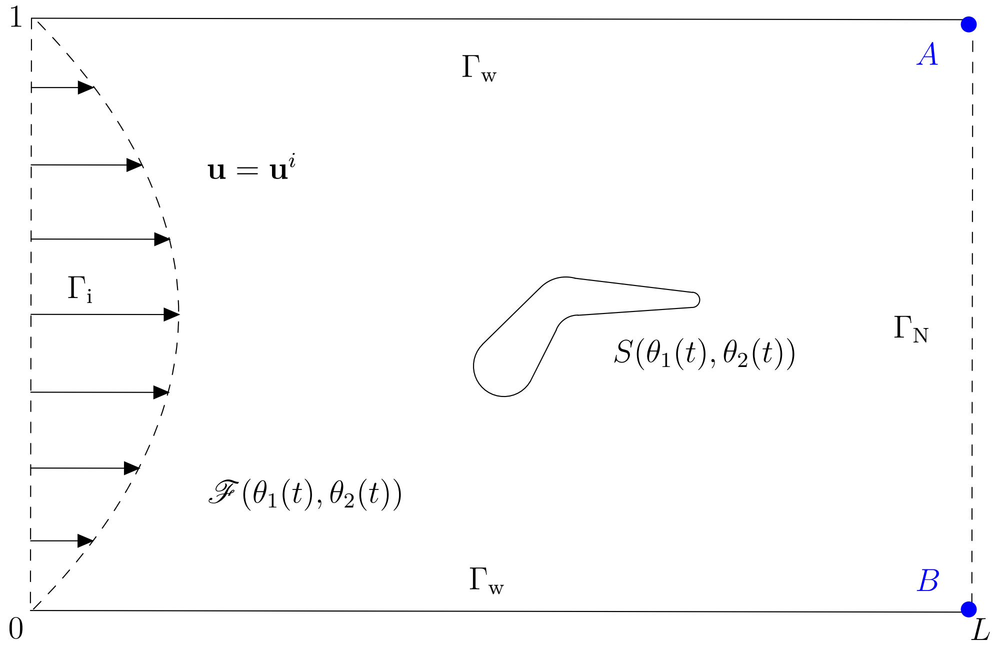


FIGURE 2.1 – The geometrical configuration.

The volume occupied by the structure depends on two parameters denoted (θ_1, θ_2), it is a closed subset of Ω that we denote $S(\theta_1, \theta_2) \subset \Omega$. The volume filled by the fluid is denoted $\mathscr{F}(\theta_1, \theta_2) = \Omega \backslash S(\theta_1, \theta_2)$.

The boundary $\partial\Omega$ can be decomposed into $\partial\Omega = \overline{\Gamma_{\rm i}} \cup \overline{\Gamma_{\rm w}} \cup \overline{\Gamma_{\rm N}}$, where $\Gamma_{\rm i} = \{0\} \times (0,1)$, $\Gamma_{\rm w} = (0, L) \times \{0,1\}$ and $\Gamma_{\rm N} = \{L\} \times (0,1)$. We also denote $\Gamma_{\rm D} = \overline{\Gamma_{\rm i}} \cup \Gamma_{\rm w}$ the part of $\partial\Omega$ where Dirichlet conditions are imposed. We now introduce the equations modelling this system.

2.1.1.1 The equations of the fluid

The velocity of the fluid is assumed to fulfil the incompressible Navier–Stokes equations

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x}) - \operatorname{div}\sigma_F(\mathbf{u}(t,\mathbf{x}), p(t,\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\ \operatorname{div}\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,\infty),\ \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)), \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\rm i}, \\ \mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\rm w}, \\ \sigma_F(\mathbf{u}(t,\mathbf{x}), p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t\in(0,\infty),\ \mathbf{x}\in\Gamma_{\rm N}, \\ \mathbf{u}(t,\mathbf{x}) = \mathbf{v}_s(t,\mathbf{x}), & t\in(0,\infty),\ \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)), \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}), \end{cases} \tag{2.1}$$

where $\mathbf{u}(t,\mathbf{x})$ and $p(t,\mathbf{x})$ are the velocity and the pressure of the fluid at point $\mathbf{x}$ and time t,

$$\sigma_F(\mathbf{u}, p) = \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) - p\mathrm{I},$$

is the Cauchy stress tensor of the fluid and $\nu > 0$ is the kinematic viscosity. The term $\mathbf{f}_{\mathscr{F}}(t,\mathbf{x})$ in $(2.1)_1$ is a force per unit mass exerted on the fluid, $\mathbf{u}^i(t,\mathbf{x})$ is a nonhomogeneous boundary datum on $\Gamma_{\rm i}$, $\mathbf{v}_s(t,\mathbf{x})$ denotes the velocity of the structure and $\mathbf{n}(\mathbf{x})$ is the outward unitary normal to Ω. Dirichlet boundary conditions are imposed on $\Gamma_{\rm D}$ and Neumann type (free output) boundary conditions are imposed on $\Gamma_{\rm N}$. We also consider an initial datum $\mathbf{u}_0(\mathbf{x})$ for the fluid velocity.

2.1.1.2 Equations of the structure

We consider that the couple of parameters (θ_1, θ_2) lies in an admissible domain $\mathbb{D}_\Theta$ which is an open connected subset of $\mathbb{R}^2$ containing $(0,0)$. We consider a function $\mathbf{X}$ defined on $\mathbb{D}_\Theta \times S(0,0)$ that computes the position of a point of the structure according to its reference position in $S(0,0)$ and the value of the parameters $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$.

Let us list below the assumptions that we make

Modelling Assumptions.

- For every $\mathbf{y} \in S(0,0)$, $\mathbf{X}(0,0,\mathbf{y}) = \mathbf{y}$. (2.2)
- For every $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1,\theta_2,S(0,0)) = S(\theta_1,\theta_2) \subset \Omega$. (2.3)
- For every $(\theta_1,\theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1,\theta_2,.)$ is a $\mathscr{C}^\infty$ diffeomorphism from $S(0,0)$ to its image. (2.4)
- The function $\mathbf{X}$ is $\mathscr{C}^\infty$ on $\mathbb{D}_\Theta \times S(0,0)$. (2.5)
- The functions $\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.)$ and $\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.)$ form a free family in $\mathbf{L}^2(\partial S(0,0))$ for every (θ_1,θ_2) in $\mathbb{D}_\Theta$. (2.6)
- No friction and no elastic energy are considered in the structure. (2.7)

More information about these assumptions can be found in Chapter 1. The inverse diffeomorphism of $\mathbf{X}(\theta_1,\theta_2,.)$, whose existence is guaranteed by (2.4), is denoted $\mathbf{Y}(\theta_1,\theta_2,.)$ and we have

$$\forall (\theta_1,\theta_2) \in \mathbb{D}_\Theta, \quad \forall \mathbf{y} \in S(0,0), \quad \mathbf{Y}(\theta_1,\theta_2,\mathbf{X}(\theta_1,\theta_2,\mathbf{y})) = \mathbf{y}. \tag{2.8}$$

The diffeomorphisms $\mathbf{X}(\theta_1,\theta_2,.)$ and $\mathbf{Y}(\theta_1,\theta_2,.)$ are illustrated in Fig. 2.2.

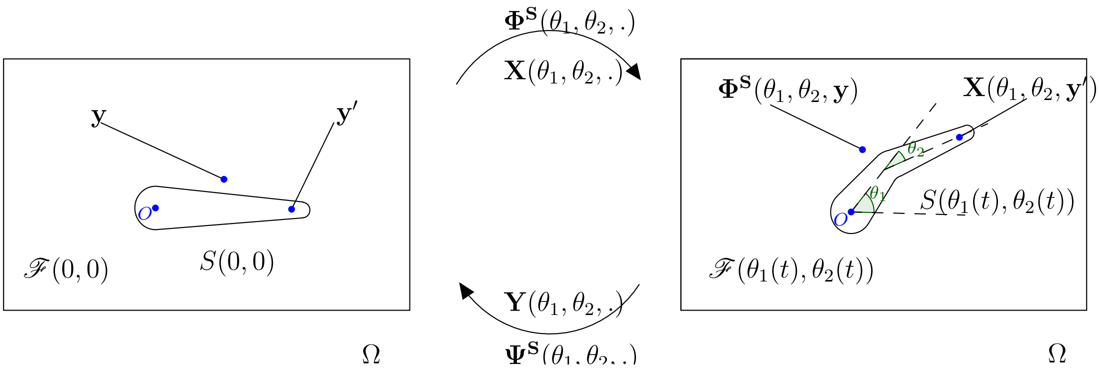


FIGURE 2.2 – Correspondence between real and reference configurations.

In the sequel, we denote $\dot{\theta}_j$ and $\ddot{\theta}_j$ the first and second time derivatives of θ_j. The equations that are satisfied by the structure read on a matricial form

$$\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \mathbf{M_I}(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2) + \mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) + \mathbf{f_s} + \mathbf{h} \quad \text{on } (0,T), \tag{2.9}$$

where $\mathbf{f_s}$ is a source term, $\mathbf{h}$ a control function,

$$\mathcal{M}_{\theta_1,\theta_2} = \begin{pmatrix} (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.))_S \\ (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.))_S \end{pmatrix} \in \mathbb{R}^{2\times 2}, \tag{2.10}$$

$$\mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) = \begin{pmatrix} -(\dot\theta_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2,.)+2\dot\theta_1\dot\theta_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2,.)+\dot\theta_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,.))_S \\ -(\dot\theta_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2,.)+2\dot\theta_1\dot\theta_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2,.)+\dot\theta_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2,.),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,.))_S \end{pmatrix} \in \mathbb{R}^2, \tag{2.11}$$

where $(.,.)_S$ is the scalar product

$$(\mathbf{\Phi},\mathbf{\Psi})_S = \int_{S(0,0)} \rho \mathbf{\Phi}(\mathbf{y}) \cdot \mathbf{\Psi}(\mathbf{y}) \,\mathrm{d}\mathbf{y}, \tag{2.12}$$

with $\rho > 0$ the mass per unit volume of the structure and

$$\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) = \begin{pmatrix} \displaystyle\int_{\partial S(\theta_1,\theta_2)} -\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}(\gamma_x)\cdot\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x \\ \displaystyle\int_{\partial S(\theta_1,\theta_2)} -\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}(\gamma_x)\cdot\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x \end{pmatrix} \in \mathbb{R}^2, \tag{2.13}$$

where $\mathbf{n}_{\theta_1,\theta_2}$ is the outward unitary normal to $\mathscr{F}(\theta_1,\theta_2)$ on $\partial S(\theta_1,\theta_2)$.

Moreover the velocity of the structure can be written

$$\forall t \in [0,\infty), \quad \forall x \in S(\theta_1(t),\theta_2(t)), \quad \mathbf{v}_s(t,\mathbf{x}) = \sum_{k=1}^{2} \dot\theta_k(t)\partial_{\theta_k}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})).$$

More information about the derivation of these equations can be found in Chapter 1.

Note that the matrix $\mathcal{M}_{\theta_1,\theta_2}$ in (2.10) is the Gram matrix of the family $(\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))$ with respect to the scalar product $(.,.)_S$. It is thus invertible due to Assumption (2.6) (if two $\mathscr{C}^\infty$ functions are not colinear in $\mathbf{L}^2(\partial S(0,0))$ then they are not colinear in $\mathbf{L}^2(S(0,0))$).

Remark 2.1.1. The proposed framework can be used to model other problems. For instance, in the case of a rigid solid whose center of mass is given by (a_1,a_2) and corresponds to $(0,0)$ in the reference configuration and whose angle of rotation is given by θ (so that three parameters are considered), the diffeomorphism $\mathbf{X}$ now depends on three parameters and is given by

$$\mathbf{X}(a_1,a_2,\theta,\mathbf{y}) = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + R_\theta\mathbf{y},$$

where $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$ and $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Moreover, we have

$$\mathcal{M}_{a_1,a_2,\theta} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \mathbf{M_I}(a_1,a_2,\theta,\dot a_1,\dot a_2,\dot\theta) = 0,$$

$$\mathbf{M_A}(a_1,a_2,\theta,\mathbf{f}) = \int_{\partial S(a_1,a_2,\theta)} \begin{pmatrix} \mathbf{f}\cdot\mathbf{e}_1 \\ \mathbf{f}\cdot\mathbf{e}_2 \\ \mathbf{f}\cdot R_{\theta+\frac{\pi}{2}}\mathbf{y} \end{pmatrix} \mathrm{d}\mathbf{x},$$

where $m = \displaystyle\int_{S(0,0,0)} \rho \,\mathrm{d}\mathbf{y}$ denotes the mass of the solid and $I = \displaystyle\int_{S(0,0,0)} \rho\mathbf{y}^2 \,\mathrm{d}\mathbf{y}$ its moment of inertia. Hence the equation (2.9) corresponds to the usual Newton's laws.

2.1.1.3 The complete set of equations

The final system that we consider is given by the following set of equations

$$
\left\{
\begin{array}{lll}
\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) + (\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x}) & & \\
\qquad\qquad -\mathrm{div}\ \sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(t,\mathbf{x}), & t\in(0,\infty), & \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathrm{div}\ \mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,\infty), & \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x}) = \displaystyle\sum_{j=1}^{2}\dot\theta_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,\infty), & \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^i(t,\mathbf{x}), & t\in(0,\infty), & \mathbf{x}\in\Gamma_{\mathrm{i}},\\
\mathbf{u}(t,\mathbf{x}) = 0, & t\in(0,\infty), & \mathbf{x}\in\Gamma_{\mathrm{w}},\\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x}) = 0, & t\in(0,\infty), & \mathbf{x}\in\Gamma_{\mathrm{N}},\\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), & & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}),\\
\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = \mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) & & \\
\qquad\qquad +\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) + \mathbf{f_s} + \mathbf{h}, & t\in(0,\infty), & \\
\theta_1(0) = \theta_{1,0}, \quad \theta_2(0) = \theta_{2,0}, & & \\
\dot\theta_1(0) = \omega_{1,0}, \quad \dot\theta_2(0) = \omega_{2,0}. & &
\end{array}
\right.
\tag{2.14}
$$

Note that the fluid domain $\mathscr{F}(\theta_1(t),\theta_2(t))$ changes over the time. The control $\mathbf{h}$ can be understood as a force acting on the structure. The data $(\theta_{1,0},\theta_{2,0})$ and $(\omega_{1,0},\omega_{2,0})$ respectively describe the initial position and velocity of the structure.

2.1.2 Statement of the main result

Existence of strong solutions to (2.14) locally in time has been proven in Chapter 1. The goal of the present study is to prove that, given a stationary state, we can choose $\mathbf{h}$ under a feedback form such that a solution to (2.14) stabilizes exponentially around that stationary state when t tends to the infinity. In this section we present our stabilization result.

The stationary state. Let $(\mathbf{w},p_{\mathbf{w}},\xi_1,\xi_2)$ be a stationary state of (2.14) associated to stationary source terms $\mathbf{f}_{\mathscr{F}}$, $\mathbf{f_s}$ and boundary datum $\mathbf{u}^i$, i.e.

$$
\left\{
\begin{array}{ll}
(\mathbf{w}(\mathbf{x})\cdot\nabla)\mathbf{w}(\mathbf{x}) - \mathrm{div}\ \sigma_F(\mathbf{w}(\mathbf{x}),p_{\mathbf{w}}(\mathbf{x})) = \mathbf{f}_{\mathscr{F}}(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\xi_1,\xi_2),\\
\mathrm{div}\ \mathbf{w}(\mathbf{x}) = 0, & \mathbf{x}\in\mathscr{F}(\xi_1,\xi_2),\\
\mathbf{w}(\mathbf{x}) = 0, & \mathbf{x}\in\partial S(\xi_1,\xi_2),\\
\mathbf{w}(\mathbf{x}) = \mathbf{u}^i(\mathbf{x}), & \mathbf{x}\in\Gamma_{\mathrm{i}},\\
\mathbf{w}(\mathbf{x}) = 0, & \mathbf{x}\in\Gamma_{\mathrm{w}},\\
\sigma_F(\mathbf{w}(\mathbf{x}),p_{\mathbf{w}}(\mathbf{x}))\mathbf{n} = 0, & \mathbf{x}\in\Gamma_{\mathrm{N}},\\
0 = \mathbf{M_I}(\xi_1,\xi_2,0,0) + \mathbf{M_A}(\xi_1,\xi_2,-\sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n}_{\xi_1,\xi_2}) + \mathbf{f_s}. &
\end{array}
\right.
\tag{2.15}
$$

Note that $\mathbf{M_I}(\xi_1,\xi_2,0,0) = 0$ and it can thus be withdrawn from (2.15).

In the sequel, we take $(\xi_1,\xi_2) = (0,0)$ to simplify the notations. This choice is not restrictive as a change of variables can bring the stationary parameters to $(0,0)$. We denote respectively $\mathscr{F}_s$ and S_s the fluid and solid domains associated to the stationary solution,

$$\mathscr{F}_s = \mathscr{F}(0,0) \qquad \text{and} \qquad S_s = S(0,0).$$

Rewriting (2.15), we consider nonhomogeneous terms $\mathbf{f}_{\mathscr{F}}$, $\mathbf{u}^i$, $\mathbf{f}_\mathbf{s}$ and a velocity–pressure profile $(\mathbf{w}, p_\mathbf{w}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ fulfilling the equations

$$\begin{cases} -\mathrm{div}\ \sigma_F(\mathbf{w}, p_\mathbf{w}) = -(\mathbf{w}\cdot\nabla)\mathbf{w} + \mathbf{f}_{\mathscr{F}} & \text{in } \mathscr{F}_s, \\ \mathrm{div}\ \mathbf{w} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{w} = 0 & \text{on } \partial S_s, \\ \mathbf{w} = \mathbf{u}^i & \text{on } \Gamma_\mathrm{i}, \\ \mathbf{w} = 0 & \text{on } \Gamma_\mathrm{w}, \\ \sigma_F(\mathbf{w}, p_\mathbf{w})\mathbf{n} = 0 & \text{on } \Gamma_\mathrm{N}, \\ (f_s)_j = \displaystyle\int_{\partial S_s} (\sigma_F(\mathbf{w}, p_\mathbf{w})\mathbf{n}_s)(\gamma_y) \cdot \partial_{\theta_j}\mathbf{X}(0,0,\gamma_y)\ \mathrm{d}\gamma_y, \end{cases} \tag{2.16}$$

where $\mathbf{n}_s$ is the outward unitary normal to $\mathscr{F}_s$ on ∂S_s,

$\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega)$ and

$$\mathbf{u}^i \in \mathbf{U}^i = \left\{ \begin{array}{ll} \mathbf{u}^i \in \mathbf{H}^{3/2}(\Gamma_\mathrm{i}) \ \text{ with } \ \mathbf{u}^i_{|\partial\Gamma_\mathrm{i}} = 0, & \displaystyle\int_0^{1/4} \frac{|\partial_{y_2} u_2^i(y_2)|^2}{y_2}\,\mathrm{d}y_2 < +\infty, \\ & \displaystyle\int_{3/4}^1 \frac{|\partial_{y_2} u_2^i(y_2)|^2}{1-y_2}\,\mathrm{d}y_2 < +\infty \end{array} \right\}. \tag{2.17}$$

More information about stationary solutions can be found in [118, Appendix].

Remark 2.1.2. The regularity of the source term $\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega)$ is used for the estimation of some nonlinear terms in Appendix D.

The diffeomorphism $\mathbf{\Phi}^\mathbf{S}$. A classical difficulty in fluid–structure problems is that the fluid domain changes over time. The classical way of getting rid of this difficulty is to use a change of variables on $\mathbf{u}$ and p in order to bring the study back into a fixed domain. This procedure uses a diffeomorphism that we have to define properly.

When the state of the structure depends only on a finite number of parameters, it is convenient to construct this diffeomorphism as an extension of the deformation of the structure into the fluid domain. The diffeomorphism used is defined as an extension of the diffeomorphism $\mathbf{X}$ given for the structure. For that reason, we use the following extension operator.

Lemma 2.1.3. *There exists a linear extension operator $\mathcal{E} : \mathbf{W}^{3,\infty}(S_s) \to \mathbf{W}^{3,\infty}(\Omega) \cap \mathbf{H}^1_0(\Omega)$ such that for every $\varphi \in \mathbf{W}^{3,\infty}(S_s)$,*

(i) $\mathcal{E}(\varphi) = \varphi$ in S_s,

(ii) $\mathcal{E}(\varphi)$ has support within $\Omega_\varepsilon = \{\mathbf{x} \in \Omega \mid \mathrm{d}(x, \partial\Omega) > \varepsilon\}$ for some $\varepsilon > 0$ such that $\mathrm{d}(S(\theta_1, \theta_2), \partial\Omega) > 2\varepsilon$ for all $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$,

(iii) $\|\varphi\|_{\mathbf{W}^{3,\infty}(\Omega)} \le C\|\varphi\|_{\mathbf{W}^{3,\infty}(S_s)}$, for some $C > 0$.

Proof. Extension results are classical, we can for instance find an extension result for smooth domains in [104, Lemma 12.2]. We can get the present result by multiplying the extension function of [104, Lemma 12.2] by a cut–off function in $\mathcal{D}(\Omega_\varepsilon)$. □

Let us denote Id the identity function, we then define the following function

$$\mathbf{\Phi}^\mathbf{S}(\theta_1, \theta_2, \mathbf{y}) = \mathbf{y} + \mathcal{E}\Big(\mathbf{X}(\theta_1, \theta_2, .) - \mathrm{Id}\Big)(\mathbf{y}), \quad \forall(\theta_1, \theta_2) \in \mathbb{D}_\Theta, \quad \forall \mathbf{y} \in \Omega. \tag{2.18}$$

We have $\nabla\mathbf{\Phi^S}(0,0,\mathbf{y}) = \mathrm{I}$, the identity matrix in $\mathbb{R}^{2\times 2}$, for every $\mathbf{y}\in\Omega$, hence $\det(\nabla\mathbf{\Phi^S}(0,0,\mathbf{y})) = 1$. Then, we can restrict $\mathbb{D}_\Theta$ such that for every $(\theta_1,\theta_2)\in\mathbb{D}_\Theta$, the function $\mathbf{\Phi^S}(\theta_1,\theta_2,.)$ is a diffeomorphism close to the identity function. We denote $\mathbf{\Psi^S}(\theta_1,\theta_2,.)$ the inverse diffeomorphism of $\mathbf{\Phi^S}(\theta_1,\theta_2,.)$

$$\forall(\theta_1,\theta_2)\in\mathbb{D}_\Theta,\quad \forall\mathbf{y}\in\Omega,\quad \mathbf{\Psi^S}(\theta_1,\theta_2,\mathbf{\Phi^S}(\theta_1,\theta_2,\mathbf{y})) = \mathbf{y}. \tag{2.19}$$

If needed, we can once more reduce $\mathbb{D}_\Theta$ to prove that $\mathbf{\Phi^S}$ and $\mathbf{\Psi^S}$ belong to $\mathscr{C}^\infty(\mathbb{D}_\Theta,\mathbf{W}^{3,\infty}(\Omega))$. These diffeomorphisms are represented in Fig. 2.2.

The properties of $\mathcal{E}$ imply that

$$\text{for every } (\theta_1,\theta_2)\in\mathbb{D}_\Theta,\ \mathbf{\Phi^S}(\theta_1,\theta_2,S_s) = S(\theta_1,\theta_2) \text{ and } \forall\mathbf{y}\in\Omega\backslash\Omega_\varepsilon,\ \mathbf{\Phi^S}(\theta_1,\theta_2,\mathbf{y}) = \mathbf{y}. \tag{2.20}$$

The stabilization problem. In order to prove a stabilization result on the nonlinear problem, we first study the linearized problem around $(\mathbf{w},p_\mathbf{w},0,0)$ and prove its stabilizability. It requires the technical hypothesis $(\boldsymbol{\mathcal{H}})_\delta$ that is presented hereafter.

In the sequel, $\mathbf{v}$ can be thought of as the difference between the state $\mathbf{u}$ and the stationary state $\mathbf{w}$ of the problem (see (2.64) for its precise definition). The linearized term in $\mathbf{v}$ in the fluid equation is the usual Oseen term $(\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}$. The linearized term in $(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)$ in the fluid equation is denoted $\mathbf{L_F}$. In the same way, we denote $\mathbf{L_S}$ the linearized term in (θ_1,θ_2) in the structure equation. Then we have

$$\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y}) = \mathbf{L}_1(\mathbf{y})\theta_1 + \mathbf{L}_2(\mathbf{y})\theta_2 + \mathbf{L}_3(\mathbf{y})\dot\theta_1 + \mathbf{L}_4(\mathbf{y})\dot\theta_2,\quad \forall\mathbf{y}\in\mathscr{F}_s, \tag{2.21}$$

and

$$\mathbf{L_S}(\theta_1,\theta_2) = \mathbf{L}_5\theta_1 + \mathbf{L}_6\theta_2, \tag{2.22}$$

where the exact expressions of the coefficients $\mathbf{L}_1 - \mathbf{L}_6$ are given in Appendix B. The coefficients $\mathbf{L}_1 - \mathbf{L}_4$ are functions and $\mathbf{L}_5 - \mathbf{L}_6$ are constant vectors of $\mathbb{R}^2$. They all depend on the non–null stationary state $(\mathbf{w},p_\mathbf{w})$ which is solution of (2.16), on the diffeomorphism $\mathbf{\Phi^S}$ and on its derivatives taken in $(\theta_1,\theta_2) = (0,0)$.

Let $\delta > 0$ be a prescribed exponential decay rate in time for the difference between the solution and the stationary state. In order to prove the main result of the study, we need the following assumption that depends on δ and corresponds to a Hautus test.

Hypothesis $(\boldsymbol{\mathcal{H}})_\delta$ (A unique continuation property)**.** Every eigenvector $(\mathbf{v},q,\theta_1,\theta_2,\omega_1,\omega_2)\in\mathbf{H}^1(\mathscr{F}_s)\times\mathrm{L}^2(\mathscr{F}_s)\times\mathbb{R}^4$ of the adjoint problem associated to the eigenvalue $\overline{\lambda}$ with $\mathcal{R}e(\overline{\lambda})\geq-\delta$, i.e. every solution of

$$\left\{\begin{array}{ll}
\operatorname{div}\sigma_F(\mathbf{v},q) - (\nabla\mathbf{w})^T\mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{v} = \overline{\lambda}\mathbf{v} & \text{in } \mathscr{F}_s,\\
\operatorname{div}\mathbf{v} = 0 & \text{in } \mathscr{F}_s,\\
\mathbf{v} = \omega_1\partial_{\theta_1}\mathbf{\Phi^S}(0,0,.) + \omega_2\partial_{\theta_2}\mathbf{\Phi^S}(0,0,.) & \text{on } \partial S_s,\\
\mathbf{v} = 0 & \text{on } \Gamma_\mathrm{D},\\
\sigma_F(\mathbf{v},q)\mathbf{n} + (\mathbf{w}\cdot\mathbf{n})\mathbf{v} = 0 & \text{on } \Gamma_\mathrm{N},\\
\displaystyle\int_{\mathscr{F}_s}\begin{pmatrix}\mathbf{L}_1(\mathbf{y})\cdot\mathbf{v}(\mathbf{y})\\ \mathbf{L}_2(\mathbf{y})\cdot\mathbf{v}(\mathbf{y})\end{pmatrix}\mathrm{d}\mathbf{y} + \begin{pmatrix}\mathbf{L}_5\cdot\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\\ \mathbf{L}_6\cdot\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\end{pmatrix} = \overline{\lambda}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, & \\
\displaystyle\int_{\mathscr{F}_s}\begin{pmatrix}\mathbf{L}_3(\mathbf{y})\cdot\mathbf{v}(\mathbf{y})\\ \mathbf{L}_4(\mathbf{y})\cdot\mathbf{v}(\mathbf{y})\end{pmatrix}\mathrm{d}\mathbf{y} - \int_{\partial S_s}\begin{pmatrix}\sigma_F(\mathbf{v},q)\mathbf{n}_s(\gamma_y)\cdot\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)\\ \sigma_F(\mathbf{v},q)\mathbf{n}_s(\gamma_y)\cdot\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)\end{pmatrix}\mathrm{d}\gamma_y + \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \overline{\lambda}\mathcal{M}_{0,0}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, &
\end{array}\right.$$

that belongs to the kernel of the adjoint of the control operator, i.e. that satisfies

$$\begin{cases} \omega_1 = 0, \\ \omega_2 = 0, \end{cases}$$

is necessarily null, i.e. $(\mathbf{v}, q, \theta_1, \theta_2, \omega_1, \omega_2) = (0, 0, 0, 0, 0, 0)$.

This hypothesis is a unique continuation property for the adjoint system. Such a property is proven for some problems, in particular for the Stokes problem with localized observation [61]. However, in our case of study, the observation is nonlocal, and to our knowledge the corresponding unique continuation property is not available in the literature. In order to lead the study of the stabilization of our problem, we assume this unique continuation property to be valid. Although we do not know how to prove it, we can reasonably think that it is generically valid. Besides, it can be checked numerically on each particular instance.

Remark 2.1.4. The hypothesis $(\boldsymbol{\mathcal{H}})_\delta$ is independent from the choice of the extension operator $\mathcal{E}$ used in Lemma 2.1.3 to construct the diffeomorphism $\mathbf{\Phi^S}$, see Appendix C.

In the sequel, $\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1, \theta_2, .)$ denotes the Jacobian matrix of $\mathbf{\Phi^S}(\theta_1, \theta_2, .)$ and $\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1, \theta_2, .))$ its cofactor matrix. The goal of the study is to prove the following theorem.

Theorem 2.1.5 (A stabilization result)**.** *Let $\delta > 0$ and assume that $(\boldsymbol{\mathcal{H}})_\delta$ is fulfilled. Let $\mathbf{f}_{\mathscr{F}} \in \mathbf{W}^{1,\infty}(\Omega)$, $\mathbf{u}^i \in \mathbf{U}^i$, $\mathbf{f_s} \in \mathbb{R}^2$, and $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ fulfilling (2.16). Then, there exists $\varepsilon > 0$ such that for every $(\mathbf{u}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}(\theta_{1,0}, \theta_{2,0})) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ satisfying the compatibility conditions*

$$\begin{cases} \mathrm{div}\ \mathbf{u}_0 = 0 & \text{in } \mathscr{F}(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0(.) = \displaystyle\sum_{j=1}^{2} \omega_{j,0} \partial_{\theta_j} \mathbf{X}(\theta_{1,0}, \theta_{2,0}, \mathbf{Y}(\theta_{1,0}, \theta_{2,0}, .)) & \text{on } \partial S(\theta_{1,0}, \theta_{2,0}), \\ \mathbf{u}_0 = \mathbf{u}^i & \text{on } \Gamma_{\mathrm{i}}, \\ \mathbf{u}_0 = 0 & \text{on } \Gamma_{\mathrm{w}}, \end{cases} \tag{2.23}$$

and

$$\|\mathbf{u}_0(\mathbf{\Phi^S}(\theta_{1,0}, \theta_{2,0}, .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \le \varepsilon,$$

there exists a control $\mathbf{h}$ given under the feedback form

$$\mathbf{h}(t) = \mathcal{K}_\delta\Big(\Big[\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1(t), \theta_2(t), .))^T \mathbf{u}(t, \mathbf{\Phi^S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}\Big], \theta_1(t), \theta_2(t), \dot\theta_1(t), \dot\theta_2(t)\Big), \tag{2.24}$$

for some linear operator $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$ such that a solution $(\mathbf{u}, p, \theta_1, \theta_2)$ to problem (1.25) *fulfils for all t in* $(0, \infty)$

$$\|\mathbf{u}(t, \mathbf{\Phi^S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot\theta_1(t)| + |\dot\theta_2(t)| \le Ce^{-\delta t},$$

for some $C > 0$ depending on the geometry, on δ and on the initial and nonhomogeneous data.

Theorem 2.1.5 is proven in Sections 2.2 and 2.3.

Remark 2.1.6. The feedback law proposed in (2.24) does not depend linearly on the state $(\mathbf{u}, p, \theta_1, \theta_2)$.

2.1.3 The functional framework

In this section we present the functional setting used in the sequel.

We denote by $\mathscr{C}^0([0,\infty);\mathbb{X})$ the set of functions that are continuous on $[0,\infty)$ and valued in $\mathbb{X}$.

Sobolev spaces. We denote $\mathrm{H}^r(\mathscr{F}_s)$ the usual Sobolev space of order $r \geq 0$. We identify $\mathrm{L}^2(\mathscr{F}_s)$ with $\mathrm{H}^0(\mathscr{F}_s)$. We will denote $\mathbf{L}^2(\mathscr{F}_s) = (\mathrm{L}^2(\mathscr{F}_s))^2$, $\mathbf{H}^r(\mathscr{F}_s) = (\mathrm{H}^r(\mathscr{F}_s))^2$ and so on.

Corners issues. The domain considered for the fluid has four corners of angle $\pi/2$. The ones that are located between Dirichlet and Neumann boundary conditions induce singularities, we denote them $A = (L,1)$ and $B = (L,0)$ (see Fig. 1.3). We also denote $\mathscr{J}_{d,n} = \{A,B\}$ the set of these corners and we define the distance of a point $\mathbf{x}$ from these corners

$$\text{for } j \in \mathscr{J}_{d,n}, \quad \text{and for } \mathbf{x} \in \Omega, \quad r_j(\mathbf{x}) = \mathrm{d}(\mathbf{x}, j). \tag{2.25}$$

Note that corners between two Dirichlet boundary conditions do not induce singularities as soon as suitable compatibility conditions are satisfied. We report to [111, Chapter 9] for more details.

Weighted Sobolev spaces. The solution to the Stokes problem in the domain with corners A and B and with a source term in $\mathbf{L}^2(\mathscr{F}_s)$ belongs to a classical Sobolev space of lower order than the one we usually have in smooth domains. In order to get the usual gain of regularity between solutions and source terms, we have to study the solution in adapted Sobolev spaces that are suitably weighted near the corners A and B. The weighted Sobolev spaces are then defined for $\beta > 0$ by

$$\mathbf{H}^2_\beta(\mathscr{F}_s) = \{\mathbf{u} \text{ with } \|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)} < +\infty\}, \qquad \mathrm{H}^1_\beta(\mathscr{F}_s) = \{p \text{ with } \|p\|_{\mathrm{H}^1_\beta(\mathscr{F}_s)} < +\infty\},$$

where the norms $\|.\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)}$ and $\|.\|_{\mathrm{H}^1_\beta(\mathscr{F}_s)}$ are given by

$$\|\mathbf{u}\|^2_{\mathbf{H}^2_\beta(\mathscr{F}_s)} = \sum_{|\alpha|=0}^{2} \sum_{i=1}^{2} \int_{\mathscr{F}_s} \left(\prod_{j \in \mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y}) \right) |\partial^\alpha u_i(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}, \tag{2.26}$$

and

$$\|p\|^2_{\mathrm{H}^1_\beta(\mathscr{F}_s)} = \sum_{|\alpha|=0}^{1} \int_{\mathscr{F}_s} \left(\prod_{j \in \mathscr{J}_{d,n}} r_j^{2\beta}(\mathbf{y}) \right) |\partial^\alpha p(\mathbf{y})|^2 \, \mathrm{d}\mathbf{y}. \tag{2.27}$$

Here the sum is on all multi–index α of length $|\alpha| \leq 2$ for (2.26), $|\alpha| \leq 1$ for (2.27) and r_j is defined in (2.25).

Steady Stokes problem with corners. The following lemma from [118] explains how and why the spaces $\mathbf{H}^2_\beta(\mathscr{F}_s)$ and $\mathrm{H}^1_\beta(\mathscr{F}_s)$ appear in the presence of corners. It gives the expected result for the steady Stokes problem in $\mathscr{F}_s$ with weigthed Sobolev spaces and the regularity obtained in the classical Sobolev spaces.

Lemma 2.1.7. *[118, Theorem 2.5.] Let us assume that $\mathbf{f}_{\mathscr{F}} \in \mathbf{L}^2(\mathscr{F}_s)$. The unique solution $(\mathbf{u}, p)$ to the Stokes problem*

$$\begin{cases} -\mathrm{div}\, \sigma_F(\mathbf{u}, p) = \mathbf{f}_{\mathscr{F}} & \text{in } \mathscr{F}_s, \\ \mathrm{div}\, \mathbf{u} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{u} = 0 & \text{on } \Gamma_{\mathrm{D}} \cup \partial S_s, \\ \sigma_F(\mathbf{u}, p)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{2.28}$$

belongs to $\mathbf{H}^2_\beta(\mathscr{F}_s) \times \mathrm{H}^1_\beta(\mathscr{F}_s)$ *for some* $\beta \in (0,1/2)$ *and to* $\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s) \times \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)$ *for some* $\varepsilon_0 \in (0,1/2)$*. Moreover, we have the following estimate*

$$\|\mathbf{u}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)\cap\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s)} + \|p\|_{\mathrm{H}^1_\beta(\mathscr{F}_s)\cap\mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)} \leq C\|\mathbf{f}_{\mathscr{F}}\|_{\mathbf{L}^2(\mathscr{F}_s)}. \tag{2.29}$$

Remark 2.1.8. A consequence of Lemma 2.1.7 is that the solution $(\mathbf{w}, p_{\mathbf{w}})$ of (2.16)–(2.17) belongs to $\mathbf{H}^2_\beta(\mathscr{F}_s) \times \mathrm{H}^1_\beta(\mathscr{F}_s)$. This is the regularity that we will use for $(\mathbf{w}, p_{\mathbf{w}})$ in the sequel. A proof of this statement can be achieved by lifting the datum $\mathbf{u}^i$, which is done in Chapter 1, Lemma 1.2.8.

Keep in mind that $\mathbf{n}_s$ is the outward unitary normal to $\mathscr{F}_s$. Note that, according to the regularity proven in Lemma 2.1.7, the traces $p_{|\partial\mathscr{F}_s}$ and $\partial_{\mathbf{n}_s}\mathbf{u}_{|\partial\mathscr{F}_s}$ are well defined, which gives a meaning to all integrations by parts.

Also note that according to [80, Theorem 1.4.3.1], there exists a continuous extension operator from $\mathbf{H}^r(\mathscr{F}_s)$ to $\mathbf{H}^r(\mathbb{R}^2)$ for every $r > 0$. This implies that all the classical Sobolev embeddings and interpolations are valid despite the presence of corners.

2.1.4 Scientific context

There are several works providing stabilization results in the context of Navier–Stokes equations. For instance, the stabilization of a viscous fluid is treated for the wake of a cylinder in [71, 77, 78, 118] and for a cavity in [115]. A first strategy used in [77, 78, 115] reposes on the Proper Orthogonal Decomposition (POD) approach. Another approach consists in constructing a feedback operator by means of a Riccati equation making the closed–loop system stable [118, 128, 129]. If needed, it is possible to use dynamical controllers to meet compatibility conditions between the fluid initial datum and the initial control value, the control is then computed as the solution of an ODE [7, 8].

When we consider fluid–structure interaction problems the same strategies can be used. The reader can refer for instance to [143] for a stabilization by a POD approach of a fluid around an airfoil. It is also possible to build a stabilizing feedback control that uses only the state of the structure, see [48] for a 1D and [148] for a 2D fluid–solid interaction problems.

In the present study, we use a stabilizing control that is given under a feedback form and uses the state of the fluid and the structure. The feedback operator is computed via the solution of a finite dimensional Riccati equation. This is helpful when treating numerical simulations which are not the point of the current paper and are a work currently in progress. The same strategy has already been used to prove stabilization of strong solutions, which is what we aim for, and more recently stabilization of weak solutions to a fluid–beam interaction problem [11]. In the literature, the feedback control can be a Dirichlet datum imposed to the fluid on some part of the boundary [9, 116], it can be a change in the shape of the structure [50, 51] or a force acting on the structure [116, 131].

Although the control that we use in the current study acts on the structure, the study [9] is the closest one from what we want to prove. It treats the stabilization of a fluid–rigid body system by a feedback control law acting on the boundary of the fluid domain. In the current study, we follow its framework and account for the deformability of the structure and the control acting on the acceleration of the structure. Additional difficulties are induced by the corners on $\partial\Omega$, more information about them can be found in [111, 118].

2.1.5 Outline of the paper

In Section 2.2, we prove the existence of a feedback control law that stabilizes the solution of the linearized system in the fixed domain $\mathscr{F}_s$ around the stationary state $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$. The proof relies on the analysis of the properties of the linearized system. In Section 2.3, we extend successively the previous result to the full nonlinear system in the fixed domain and in the moving domain. The former is proven via a fixed point argument and the latter uses a change of variables. The linearized terms are summed up in Appendix B. An idea of the proof of Remark 2.1.4 is given in Appendix C. The proof of the technical estimates of the nonlinear terms can be found in Appendix D.

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2.2 Stabilization of the linearized problem

In the whole Section 2.2 we consider stationary nonhomogeneous terms $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f_s}) \in \mathbf{W}^{1,\infty}(\Omega)\times \mathbf{U}^i \times \mathbb{R}^2$ and a stationary state $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^2_\beta(\mathscr{F}_s) \times \mathrm{H}^1_\beta(\mathscr{F}_s)$ that fulfil (2.16). Our goal is to find a control law $\mathbf{h}$ under a feedback form that stabilizes the linearized problem with a given exponential decay δ in time.

2.2.1 The linearized problem

We study the linearized system associated to (2.14). The variables $(\mathbf{v}, q)$ correspond roughly to the difference between $(\mathbf{u}, p)$ written in the fixed domain $\mathscr{F}_s$ and $(\mathbf{w}, p_{\mathbf{w}})$. A proper definition of these variables can be found in (2.64). Here is the linearized system around $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$

$$\begin{cases}
\dfrac{\partial \mathbf{v}}{\partial t}+(\mathbf{w}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{w}-\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y})-\nu\Delta\mathbf{v}+\nabla q=\mathbf{f} & \text{in } (0,\infty)\times\mathscr{F}_s,\\
\operatorname{div}\mathbf{v}=0 & \text{in } (0,\infty)\times\mathscr{F}_s,\\
\mathbf{v}=\dot\theta_1\partial_{\theta_1}\mathbf{\Phi^S}(0,0,.)+\dot\theta_2\partial_{\theta_2}\mathbf{\Phi^S}(0,0,.)+\mathbf{g} & \text{on } (0,\infty)\times\partial S_s,\\
\mathbf{v}=0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{D}},\\
\sigma_F(\mathbf{v},q)\mathbf{n}=0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{N}},\\
\mathbf{v}(0,.)=\mathbf{v}_0 & \text{in } \mathscr{F}_s,\\
\mathcal{M}_{0,0}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix}=\begin{pmatrix}\displaystyle\int_{\partial S_s}[q\mathrm{I}-\nu(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)]\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\\ \displaystyle\int_{\partial S_s}[q\mathrm{I}-\nu(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)]\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\end{pmatrix} & \\
\qquad\qquad +\mathbf{L_S}(\theta_1,\theta_2)+\mathbf{s}+\mathbf{h} & \text{on } (0,\infty),\\
\theta_1(0)=\theta_{1,0},\qquad \theta_2(0)=\theta_{2,0}, & \\
\dot\theta_1(0)=\omega_{1,0},\qquad \dot\theta_2(0)=\omega_{2,0}, &
\end{cases} \tag{2.30}$$

where $(\mathbf{f}, \mathbf{g}, \mathbf{s})$ are nonhomogeneous terms and $\mathbf{v}_0$ an initial datum for $\mathbf{v}$. Here, $\mathbf{L_F} \in \mathbf{L}^2(\mathscr{F}_s, \mathcal{L}(\mathbb{R}^4, \mathbb{R}^2))$ and $\mathbf{L_S} \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ depend on the stationary state $(\mathbf{w}, p_{\mathbf{w}})$ and on the diffeomorphism $\mathbf{\Phi^S}$, they are given by

$$\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y}) = \mathbf{L}_1(\mathbf{y})\theta_1+\mathbf{L}_2(\mathbf{y})\theta_2+\mathbf{L}_3(\mathbf{y})\dot\theta_1+\mathbf{L}_4(\mathbf{y})\dot\theta_2, \quad \forall \mathbf{y}\in\mathscr{F}_s, \tag{2.31}$$

$$\mathbf{L_S}(\theta_1,\theta_2) = \mathbf{L}_5\theta_1+\mathbf{L}_6\theta_2, \tag{2.32}$$

where the coefficients $\mathbf{L}_1$–$\mathbf{L}_6$ are properly defined in Appendix B ((B.1)–(B.6)).

For any fixed $\delta > 0$ such that $(\boldsymbol{\mathcal{H}})_\delta$ holds, we use the following spaces

$$\mathbb{U}_\delta^\infty = \{\mathbf{v} \text{ with } e^{\delta t}\mathbf{v} \in \mathrm{L}^2(0,\infty;\mathbf{H}^2_\beta(\mathscr{F}_s)) \cap \mathscr{C}^0([0,\infty);\mathbf{H}^1(\mathscr{F}_s)) \cap \mathrm{H}^1(0,\infty;\mathbf{L}^2(\mathscr{F}_s))\}, \tag{2.33}$$
$$\mathbb{P}_\delta^\infty = \{ q \text{ with } e^{\delta t}q \in \mathrm{L}^2(0,\infty;\mathrm{H}^1_\beta(\mathscr{F}_s)) \}, \tag{2.34}$$
$$\Theta_\delta^\infty = \{ (\theta_1,\theta_2) \text{ with } e^{\delta t}(\theta_1,\theta_2) \in \mathrm{H}^2(0,\infty;\mathbb{R}^2) \}, \tag{2.35}$$
$$\mathbb{F}_\delta^\infty = \{ \mathbf{f} \text{ with } e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0,\infty;\mathbf{L}^2(\mathscr{F}_s)) \}, \tag{2.36}$$
$$\mathbb{G}_\delta^\infty = \{ \mathbf{g} \text{ with } e^{\delta t}\mathbf{g} \in \mathrm{H}^1(0,\infty;\mathbf{H}^{3/2}(\partial S_s)) \}, \tag{2.37}$$
$$\mathbb{S}_\delta^\infty = \{ \mathbf{s} \text{ with } e^{\delta t}\mathbf{s} \in \mathrm{L}^2(0,\infty;\mathbb{R}^2) \}. \tag{2.38}$$

All these spaces are equipped with their natural norms, e.g. for Θ_δ^∞,

$$\|(\theta_1,\theta_2)\|_{\Theta_\delta^\infty} = \|(\theta_1,\theta_2)e^{\delta t}\|_{\mathrm{H}^2(0,\infty;\mathbb{R}^2)}.$$

The goal of Section 2.2 is to prove the following result.

Proposition 2.2.1. *Let $\delta > 0$ and assume that $(\boldsymbol{\mathcal{H}})_\delta$ is fulfilled. There exists a feedback operator $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$, such that for every $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4$, $\mathbf{f} \in \mathbb{F}_\delta^\infty$, $\mathbf{g} \in \mathbb{G}_\delta^\infty$ and $\mathbf{s} \in \mathbb{S}_\delta^\infty$ fulfilling the compatibility conditions*

$$\begin{cases} \operatorname{div} \mathbf{v}_0 = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v}_0 = \displaystyle\sum_{j=1}^2 \omega_{j,0}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,.) + \mathbf{g}(0) & \text{on } \partial S_s, \\ \mathbf{v}_0 = 0 & \text{on } \Gamma_{\mathrm{D}}, \end{cases} \tag{2.39}$$

problem (2.30) *with the control taken as $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ admits a unique solution $(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ with the following estimate,*

$$\begin{aligned} \|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1,\theta_2)\|_{\Theta_\delta^\infty} \le C(&\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \\ &+\|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty}), \end{aligned} \tag{2.40}$$

where C does not depend on the initial conditions and on the source terms.
A consequence is that for every $t \in (0,\infty)$,

$$\begin{aligned} \|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot\theta_1(t)| + |\dot\theta_2(t)| \le C\Big(&\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| \\ &+ |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty}\Big)e^{-\delta t}. \end{aligned} \tag{2.41}$$

We first work on the homogeneous system associated to (2.30). In Section 2.2.2 we develop the functional framework used to write the homogeneous system under a semigroup formulation. In Section 2.2.3 we study the adjoint operator. In Section 2.2.4 we exhibit a feedback operator $\mathcal{K}_\delta$ that stabilizes the homogeneous problem. We then prove Proposition 2.2.1 in Section 2.2.5.

2.2.2 Functional framework for the semigroup formulation

In this section, the linear problem considered in Section 2.2.1 is rewritten under a semigroup formulation and closely follows Chapter 1. This enables us to use the classical strategy to derive a feedback operator $\mathcal{K}_\delta$ stabilizing the system (2.30) (see [24, Part V] and [54, Section 5.2] for a full presentation of this method). We use the spaces

$$\mathbb{H} = \left\{ \begin{array}{r} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \quad \mathrm{div}\, \mathbf{v} = 0 \text{ in } \mathscr{F}_s, \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{D}}, \\ \mathbf{v} \cdot \mathbf{n}_s = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,.) \cdot \mathbf{n}_s \text{ on } \partial S_s \end{array} \right\}, \tag{2.42}$$

and

$$\mathbb{V} = \left\{ \begin{array}{r} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4, \quad \mathrm{div}\, \mathbf{v} = 0 \text{ in } \mathscr{F}_s, \quad \mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{D}}, \\ \mathbf{v} = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,.) \text{ on } \partial S_s \end{array} \right\}. \tag{2.43}$$

The spaces $\mathbb{H}$ and $\mathbb{V}$ are respectively endowed with the scalar products $(.,.)_0$ of $\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ and $(.,.)_1$ of $\mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4$ defined by

$$\forall (\mathbf{v}^j, \theta_1^j, \theta_2^j, \omega_1^j, \omega_2^j) \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4,$$

$$\Big((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a), (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \Big)_0 = \int_{\mathscr{F}_s} \mathbf{v}^a \cdot \mathbf{v}^b \, \mathrm{d}\mathbf{y} + \sum_j \theta_j^a \theta_j^b + (\omega_1^a \ \omega_2^a) \mathcal{M}_{0,0} \begin{pmatrix} \omega_1^b \\ \omega_2^b \end{pmatrix},$$

and

$$\forall (\mathbf{v}^j, \theta_1^j, \theta_2^j, \omega_1^j, \omega_2^j) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{R}^4,$$

$$\Big((\mathbf{v}^a, \theta_1^a, \theta_2^a, \omega_1^a, \omega_2^a), (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b) \Big)_1 = \int_{\mathscr{F}_s} (\mathbf{v}^a \cdot \mathbf{v}^b + \nabla \mathbf{v}^a : \nabla \mathbf{v}^b) \, \mathrm{d}\mathbf{y} + \sum_j \theta_j^a \theta_j^b + (\omega_1^a \ \omega_2^a) \mathcal{M}_{0,0} \begin{pmatrix} \omega_1^b \\ \omega_2^b \end{pmatrix},$$

where $\mathcal{M}_{0,0}$ is defined in (2.10) for $\theta_1 = \theta_2 = 0$.

In the sequel, for a better readability, we use the notation

$$(f_j)_{j=1,2} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Lemma 2.2.2. *The orthogonal space to* $\mathbb{H}$ *with respect to the scalar product* $(.,.)_0$ *is*

$$(\mathbb{H})^\perp = \left\{ \left(\nabla p, 0, 0, -\mathcal{M}_{0,0}^{-1} \left(\int_{\partial S_s} p \mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \right)_{j=1,2} \right) \text{ with } \ p \in \mathrm{H}^1(\mathscr{F}_s), \ p = 0 \text{ on } \Gamma_{\mathrm{N}} \right\}.$$

Proof. See Chapter 1, Lemma 1.2.3. □

We adapt Rellich's compact embedding Theorem to our functional framework.

Lemma 2.2.3. *The embedding from* $\mathbb{V}$ *into* $\mathbb{H}$ *is compact.*

Proof. The proof is an easy consequence of Rellich's compact embedding Theorem [80, Theorem 1.4.3.2.]. □

We define the operator $(A, D(A))$ on $\mathbb{H}$ by

$$D(A) = \left\{ \begin{array}{r} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, \mathbf{v} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s), \ \exists q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s) \text{ such that} \\ \mathrm{div}\, \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\}, \tag{2.44}$$

$$A \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \mathrm{div}\, \sigma_F(\mathbf{v}, q) + \mathbf{L}_{\mathbf{F}}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v} \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1} \left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \right)_{j=1,2} + \mathbf{L}_{\mathbf{S}}(\theta_1, \theta_2) \right) \end{pmatrix}, \tag{2.45}$$

where ε_0 is introduced in Lemma 2.1.7 and $\Pi_{\mathbb{H}}$ denotes the orthogonal projection of $\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4$ onto $\mathbb{H}$. The next lemmas state some properties of $(A, D(A))$.

Remark 2.2.4. The use of q in the definition of $(A, D(A))$ is useful to guarantee that $\operatorname{div}\sigma_F(\mathbf{v}, q)$ belongs to $\mathbf{L}^2(\mathscr{F}_s)$ and then that the application of $\Pi_{\mathbb{H}}$ in the right hand-side of (2.45) makes sense.

Lemma 2.2.5. *The operator A is uniquely defined.*

Proof. A similar proof is presented in Chapter 1, Lemma 1.2.4, we need to slightly adapt it in order to take into account the terms coming from the linearization around a stationary solution.

Let $(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in D(A)$ and consider two functions p, $q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)$ satisfying the conditions appearing into the definition of $D(A)$. Then, $\operatorname{div}\,\sigma_F(0, p-q) = -\nabla(p-q) \in \mathbf{L}^2(\mathscr{F}_s)$ implies $p - q \in \mathrm{H}^1(\mathscr{F}_s)$, and $\sigma_F(0, p-q)\mathbf{n} = 0$ on Γ_N implies $p - q = 0$ on Γ_N.

Now,

$$\begin{pmatrix} \operatorname{div}\sigma_F(\mathbf{v}, p) + \mathbf{L_F}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{v} \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1}\left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, p)\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{\Phi^S}(0,0,\gamma_y)\mathrm{d}\gamma_y\right)_{j=1,2} + \mathbf{L_S}(\theta_1, \theta_2)\right) \end{pmatrix}$$
$$- \begin{pmatrix} \operatorname{div}\sigma_F(\mathbf{v}, q) + \mathbf{L_F}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) - (\mathbf{v}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{v} \\ \omega_1 \\ \omega_2 \\ \mathcal{M}_{0,0}^{-1}\left(\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{\Phi^S}(0,0,\gamma_y)\mathrm{d}\gamma_y\right)_{j=1,2} + \mathbf{L_S}(\theta_1, \theta_2)\right) \end{pmatrix}$$
$$= \begin{pmatrix} \nabla(p-q) \\ 0 \\ 0 \\ -\mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_s} (p-q)\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{\Phi^S}(0,0,\gamma_y)\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix},$$

which belongs to $\mathbb{H}^\perp$ according to Lemma 2.2.2. Therefore A is uniquely defined. □

Before going further, let us point out that $D(A)$ can be characterized as follows.

Lemma 2.2.6. *We have*

$$D(A) = \left\{ \begin{array}{r} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, \mathbf{v} \in \mathbf{H}^2_\beta(\mathscr{F}_s),\ \exists q \in \mathrm{H}^1_\beta(\mathscr{F}_s) \text{ such that} \\ \operatorname{div}\,\sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 \text{ on } \Gamma_\mathrm{N} \end{array} \right\}.$$

Proof. Assume that $(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2)$ belongs to $D(A)$ given by (2.44). Then $(\mathbf{v}, q)$ satisfies

$$\begin{cases} \operatorname{div}\,\sigma_F(\mathbf{v}, q) & \in \mathbf{L}^2(\mathscr{F}_s), \\ \operatorname{div}\,\mathbf{v} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v} = 0 & \text{on } \Gamma_\mathrm{D}, \\ \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 & \text{on } \Gamma_\mathrm{N}, \\ \mathbf{v} = \sum_j \omega_j \partial_{\theta_j}\mathbf{\Phi^S}(0,0,.) & \text{on } \partial S_s. \end{cases}$$

According to [118, Theorem 2.16], there exists $\mathbf{v}_s \in \mathbf{H}^2(\mathscr{F}_s)$ such that

$$\begin{cases} \operatorname{div}\,\sigma_F(\mathbf{v}_s, 0) = 0 & \text{in } \mathscr{F}_s \\ \operatorname{div}\,\mathbf{v}_s = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v}_s = 0 & \text{on } \Gamma_\mathrm{D}, \\ \sigma_F(\mathbf{v}_s, 0)\mathbf{n} = 0 & \text{on } \Gamma_\mathrm{N}, \\ \mathbf{v}_s = \sum_j \omega_j \partial_{\theta_j}\mathbf{\Phi^S}(0,0,.) & \text{on } \partial S_s. \end{cases}$$

Let $\mathbf{f} = -\text{div}\,\sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s)$. Then $(\mathbf{v}-\mathbf{v}_s, q)$ satisfies

$$\begin{cases} -\text{div}\,\sigma_F(\mathbf{v}-\mathbf{v}_s, q) = \mathbf{f} & \text{in } \mathscr{F}_s, \\ \text{div}\,(\mathbf{v}-\mathbf{v}_s) = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v}-\mathbf{v}_s = 0 & \text{on } \Gamma_{\rm D} \cup \partial S_s, \\ \sigma_F(\mathbf{v}-\mathbf{v}_s, q)\mathbf{n} = 0 & \text{on } \Gamma_{\rm N}. \end{cases}$$

According to Lemma 2.1.7, $\mathbf{v}-\mathbf{v}_s \in \mathbf{H}^2_\beta(\mathscr{F}_s) \cap \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s)$, $q \in \mathrm{H}^1_\beta(\mathscr{F}_s) \cap \mathbf{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)$. This ends the proof. □

We define the bilinear form a_1 on $\mathbb{V}\times\mathbb{V}$ for every $(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2)$ and $(\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)$ in $\mathbb{V}$ by

$$\begin{aligned} \mathrm{a}_1((\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2), (\mathbf{v}^b, \theta_1^b, \theta_2^b, \omega_1^b, \omega_2^b)) = \frac{\nu}{2}\int_{\mathscr{F}_s} (\nabla\mathbf{v} + \nabla\mathbf{v}^T) : (\nabla\mathbf{v}^b + (\nabla\mathbf{v}^b)^T)\,\mathrm{d}\mathbf{y} \\ + \int_{\mathscr{F}_s} \Big((\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}\Big)\cdot\mathbf{v}^b\,\mathrm{d}\mathbf{y}. \end{aligned} \tag{2.46}$$

We define the operator $(A_1, D(A_1))$ on $\mathbb{H}$ by

$$D(A_1) = \{\mathbf{z} \in \mathbb{V} \ \text{ with } \ \tilde{\mathbf{z}} \mapsto \mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) \text{ is } \mathbb{H}\text{−continuous }\},$$

and

$$\forall \mathbf{z} \in D(A_1),\ \forall \tilde{\mathbf{z}} \in \mathbb{V},\ (A_1\mathbf{z}, \tilde{\mathbf{z}})_0 = -\mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}).$$

Lemma 2.2.7. *We have*

$$D(A_1) = D(A),$$

and

$$A_1 \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \text{div}\,\sigma_F(\mathbf{v}, q) - (\mathbf{v}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{v} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix}.$$

Proof. The inclusion $D(A) \subset D(A_1)$ comes easily. Moreover, for every $\mathbf{z} = (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in D(A)$, an integration by parts yields

$$\forall \tilde{\mathbf{z}} \in \mathbb{V}, \quad (A_1\mathbf{z}, \tilde{\mathbf{z}})_0 = \left(\begin{pmatrix} \text{div}\,\sigma_F(\mathbf{v}, q) - (\mathbf{v}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{v} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\left(\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\right)_{j=1,2} \end{pmatrix}, \tilde{\mathbf{z}} \right)_0.$$

Let us now prove the reverse inclusion $D(A_1) \subset D(A)$. Let $\mathbf{z} \in D(A_1)$. According to Riesz representation theorem, there exists $\mathbf{f} \in \mathbb{H}$ such that

$$\forall \tilde{\mathbf{z}} \in \mathbb{V}, \quad \mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) = (\mathbf{f}, \tilde{\mathbf{z}})_0.$$

We write $\mathbf{f} = (\mathbf{f}_{\mathbf{v}}, \mathbf{f}_{\theta_1}, \mathbf{f}_{\theta_2}, \mathbf{f}_{\omega_1}, \mathbf{f}_{\omega_2})$. For $\tilde{\mathbf{v}} \in \mathcal{D}_{\rm div} = \{\mathbf{u} \in (\mathscr{C}_c^\infty(\mathscr{F}_s))^2 \ \text{ with } \ \text{div}\,\mathbf{u} = 0\}$, we know that $(\tilde{\mathbf{v}}, 0, 0, 0, 0)$ belongs to $\mathbb{V}$, and an integration by parts yields

$$\mathrm{a}_1(\mathbf{z}, (\tilde{\mathbf{v}}, 0, 0, 0, 0)) = \int_{\mathscr{F}_s} (-\text{div}\,\sigma_F(\mathbf{v}, 0) + (\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v})\cdot\tilde{\mathbf{v}}\,\mathrm{d}\mathbf{y} = \int_{\mathscr{F}_s} \mathbf{f}_{\mathbf{v}}\cdot\tilde{\mathbf{v}}\,\mathrm{d}\mathbf{y}.$$

Then, according to [146, Lemma 2.2.2], there exists $\widehat{q} \in \mathrm{L}^2(\mathscr{F}_s)$ such that

$$-\mathrm{div}\ \sigma_F(\mathbf{v}, \widehat{q}) + (\mathbf{v} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v} = \mathbf{f_v} \text{ in } \mathscr{F}_s, \tag{2.47}$$

and thus div $\sigma_F(\mathbf{v}, \widehat{q})$ belongs to $\mathbf{L}^2(\mathscr{F}_s)$, which gives a meaning to $\sigma_F(\mathbf{v}, \widehat{q})\mathbf{n}_s$ on $\partial\mathscr{F}_s$.

Now, let us prove that $\sigma_F(\mathbf{v}, \widehat{q})\mathbf{n}$ is constant along Γ_{N}. Let $\mathbf{g} \in (\mathscr{C}_c^\infty(\Gamma_{\mathrm{N}}))^2$ fulfilling $\displaystyle\int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{n}\, \mathrm{d}\gamma_y = 0$. According to [72, Theorem IV.1.1], there exists $\mathbf{v_g} \in \mathbf{H}^1(\mathscr{F}_s)$ satisfying

$$\begin{cases} \mathrm{div}\ \mathbf{v_g} = 0 \text{ in } \mathscr{F}_s, \\ \mathbf{v_g} = 0 \text{ on } \Gamma_{\mathrm{D}} \cup \partial S_s, \\ \mathbf{v_g} = \mathbf{g} \text{ on } \Gamma_{\mathrm{N}}. \end{cases}$$

We know that $(\mathbf{v_g}, 0, 0, 0, 0)$ belongs to $\mathbb{V}$. An integration by parts yields

$$\begin{aligned} \mathrm{a}_1(\mathbf{z}, (\mathbf{v_g}, 0, 0, 0, 0)) = \int_{\mathscr{F}_s} &(-\mathrm{div}\ \sigma_F(\mathbf{v}, \widehat{q}) + (\mathbf{v} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v}) \cdot \mathbf{v_g}\, \mathrm{d}\mathbf{y} \\ &+ \int_{\Gamma_{\mathrm{N}}} \sigma_F(\mathbf{v}, \widehat{q})\mathbf{n} \cdot \mathbf{g}\, \mathrm{d}\gamma_y = \int_{\mathscr{F}_s} \mathbf{f_v} \cdot \mathbf{v_g}\, \mathrm{d}\mathbf{y}, \end{aligned}$$

and with (2.47) we get

$$\int_{\Gamma_{\mathrm{N}}} \sigma_F(\mathbf{v}, \widehat{q})\mathbf{n} \cdot \mathbf{g}\, \mathrm{d}\gamma_y = 0.$$

The previous equality holds for every $\mathbf{g} \in (\mathscr{C}_c^\infty(\Gamma_{\mathrm{N}}))^2$ fulfilling $\displaystyle\int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{n}\, \mathrm{d}\gamma_y = 0$, then there exists a constant c such that $\sigma_F(\mathbf{v}, \widehat{q})\mathbf{n} = c\, \mathbf{n}$ on Γ_{N}.

Let $q = \widehat{q} - c \in \mathrm{L}^2(\mathscr{F}_s)$, we have div $\sigma_F(\mathbf{v}, q) =$ div $\sigma_F(\mathbf{v}, \widehat{q})$ and $\sigma_F(\mathbf{v}, q)\mathbf{n} = 0$ on Γ_{N}. Moreover, $(\mathbf{v}, q)$ satisfies

$$\begin{cases} \mathrm{div}\ \sigma_F(\mathbf{v}, q) & \in \mathbf{L}^2(\mathscr{F}_s), \\ \mathrm{div}\ \mathbf{v} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{v} = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \mathbf{v} = \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi^S}(0, 0, .) & \text{on } \partial S_s, \\ \sigma_F(\mathbf{v}, q)\mathbf{n} = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{cases}$$

We finish this proof with a lifting of the boundary datum on ∂S_s and Lemma 2.1.7, like in the proof of Lemma 2.2.6. We get $D(A_1) \subset D(A)$, thus concluding the proof of Lemma 2.2.7. □

Lemma 2.2.8. *The operator A generates an analytic semigroup on $\mathbb{H}$ and has compact resolvent.*

Proof. According to [118, p. 3015], there exists a constant $c > 0$ such that for $\lambda > 0$ large enough, we have

$$\forall \mathbf{z} \in \mathbb{V}, \qquad \mathrm{a}_1(\mathbf{z}, \mathbf{z}) + \lambda \|\mathbf{z}\|_{\mathbb{H}}^2 \geq c\|\mathbf{z}\|_{\mathbb{V}}^2. \tag{2.48}$$

Moreover, according to [24, Theorem 2.12, p. 115], the estimate (2.48) implies that the operator A_1 generates an analytic semigroup on $\mathbb{H}$.

Now, as $A - A_1 \in \mathcal{L}(\mathbb{H})$, according to [125, Corollary 2.2], A generates an analytic semigroup on $\mathbb{H}$.

We have $D(A) \subset \mathbb{V}$ and, according to Lemma 2.2.3, the embedding from $\mathbb{V}$ into $\mathbb{H}$ is compact. The operator A then has compact resolvent. This ends this proof. □

2.2.3 Study of the adjoint operator

In order to simplify the notations, in the sequel we do not write d$\mathbf{y}$ or dγ_y anymore in the integrals.

Lemma 2.2.9. *The adjoint operator A_1^* of A_1 with respect to the scalar product $(.,.)_0$ is given by*

$$D(A_1^*) = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, & \mathbf{v} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s), \quad \exists q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s) \text{ such that} \\ & \operatorname{div} \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} + (\mathbf{w}\cdot\mathbf{n})\mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\},$$

and

$$A_1^* \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\mathbf{v}, q) - (\nabla \mathbf{w})^T \mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{v} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1} \left(\displaystyle\int_{\partial S_s} -\sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \right)_{j=1,2} \end{pmatrix}.$$

Proof. The adjoint A_1^* of A_1 is defined by

$$D(A_1^*) = \{ \ \tilde{\mathbf{z}} \in \mathbb{V} \ \text{ with } \ \mathbf{z} \mapsto \mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) \text{ is } \mathbb{H}\text{–continuous } \},$$

and

$$\forall \mathbf{z} \in \mathbb{V}, \quad \forall \tilde{\mathbf{z}} \in D(A_1^*), \quad (\mathbf{z}, A_1^* \tilde{\mathbf{z}})_0 = -\mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}).$$

Let us denote

$$E = \left\{ \begin{array}{ll} (\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) \in \mathbb{V}, & \mathbf{v} \in \mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s), \quad \exists q \in \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s) \text{ such that} \\ & \operatorname{div} \sigma_F(\mathbf{v}, q) \in \mathbf{L}^2(\mathscr{F}_s) \text{ and } \sigma_F(\mathbf{v}, q)\mathbf{n} + (\mathbf{w}\cdot\mathbf{n})\mathbf{v} = 0 \text{ on } \Gamma_{\mathrm{N}} \end{array} \right\}.$$

For every $\tilde{\mathbf{z}} = (\tilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2) \in E$, we have

$$\forall \mathbf{z} \in \mathbb{V}, \quad -\mathrm{a}_1(\mathbf{z}, \tilde{\mathbf{z}}) = \left(\begin{pmatrix} \operatorname{div} \sigma_F(\tilde{\mathbf{v}}, \widetilde{q}) - (\nabla \mathbf{w})^T \tilde{\mathbf{v}} + (\mathbf{w}\cdot\nabla)\tilde{\mathbf{v}} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1} \left(\displaystyle\int_{\partial S_s} -\sigma_F(\tilde{\mathbf{v}}, \widetilde{q})\mathbf{n}_s \cdot \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \right)_{j=1,2} \end{pmatrix}, \mathbf{z} \right)_0.$$

Then, we have $E \subset D(A_1^*)$.

To prove the reverse inclusion, the same arguments as in the proof of Lemma 2.2.7 lead to the existence of $\widetilde{q} \in \mathrm{L}^2(\mathscr{F}_s)$ such that

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma_F(\tilde{\mathbf{v}}, \widetilde{q}) & \in \mathbf{L}^2(\mathscr{F}_s), \\ \operatorname{div} \tilde{\mathbf{v}} = 0 & \text{in } \mathscr{F}_s, \\ \tilde{\mathbf{v}} = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \tilde{\mathbf{v}} = \sum\limits_j \widetilde{\omega}_j \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,.) & \text{on } \partial S_s, \\ \sigma_F(\tilde{\mathbf{v}}, \widetilde{q})\mathbf{n} + (\mathbf{w}\cdot\mathbf{n})\tilde{\mathbf{v}} = 0 & \text{on } \Gamma_{\mathrm{N}}. \end{array} \right.$$

Then $\sigma_F(\tilde{\mathbf{v}}, \widetilde{q})\mathbf{n}$ belongs to $\mathbf{H}^{1/2}(\Gamma_{\mathrm{N}})$, this implies that $(\tilde{\mathbf{v}}, \widetilde{q})$ belongs to $\mathbf{H}^{3/2+\varepsilon_0}(\mathscr{F}_s) \times \mathrm{H}^{1/2+\varepsilon_0}(\mathscr{F}_s)$. This regularity is a consequence of [111, Theorem 9.4.5] and the arguments used in the proof of [118, Theorem 2.5]. This ends the proof. □

Lemma 2.2.10. *The adjoint of A with respect to $(.,.)_0$ is given by*

$$D(A^*) = D(A_1^*),$$

and

$$A^* \begin{pmatrix} \mathbf{v} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \Pi_{\mathbb{H}} \begin{pmatrix} \operatorname{div} \sigma_F(\mathbf{v}, q) + (\mathbf{w} \cdot \nabla)\mathbf{v} - (\nabla \mathbf{w})^T \mathbf{v} \\ \displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_1 \cdot \mathbf{v} \\ \mathbf{L}_2 \cdot \mathbf{v} \end{pmatrix} + \left(\mathbf{L}_{k+4} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right)_{k=1,2} \\ \mathcal{M}_{0,0}^{-1} \left[\displaystyle\int_{\mathscr{F}_s} \begin{pmatrix} \mathbf{L}_3 \cdot \mathbf{v} \\ \mathbf{L}_4 \cdot \mathbf{v} \end{pmatrix} - \int_{\partial S_s} \begin{pmatrix} \sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_1} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \\ \sigma_F(\mathbf{v}, q)\mathbf{n}_s \cdot \partial_{\theta_2} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \end{pmatrix} + \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \right] \end{pmatrix}.$$

Proof. We have $A - A_1 \in \mathcal{L}(\mathbb{H})$, then $A^* - A_1^* \in \mathcal{L}(\mathbb{H})$ and $D(A^*) = D(A_1^*)$.

A computation of $(A\mathbf{z}, \tilde{\mathbf{z}})_0$ for $\mathbf{z} \in D(A)$ and $\tilde{\mathbf{z}} \in D(A^*)$ gives the explicit expression of A^*. □

2.2.4 Construction of a feedback operator

We define the control operator $B \in \mathcal{L}(\mathbb{R}^2, \mathbb{H})$ by

$$B\mathbf{h} = \Pi_{\mathbb{H}}(0, 0, 0, \mathcal{M}_{0,0}^{-1}\mathbf{h}). \tag{2.49}$$

The linear system (2.30) with no source terms ($\mathbf{f} = 0,\ \mathbf{g} = 0,\ \mathbf{s} = 0$) can be rewritten under the form

$$\begin{cases} \mathbf{z}'(t) = A\mathbf{z}(t) + B\mathbf{h}(t), \quad t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{2.50}$$

where $\mathbf{z} = (\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ and $\mathbf{z}_0 = (\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$.

We want to exhibit a control $\mathbf{h}$ under a feedback form that stabilizes problem (2.50). In order to guarantee a decay rate $\delta > 0$ of the solution to this problem, we consider the stabilization of $\mathbf{z}_\delta = e^{\delta t}\mathbf{z}$, solution to the problem

$$\begin{cases} \mathbf{z}_\delta'(t) = (A + \delta \mathrm{I})\mathbf{z}_\delta(t) + B\mathbf{h}_\delta(t), \quad t > 0, \\ \mathbf{z}_\delta(0) = \mathbf{z}_0, \end{cases} \tag{2.51}$$

where

$$\mathbf{h}_\delta(t) = e^{\delta t}\mathbf{h}(t), \qquad t > 0.$$

Our goal is to find a control $\mathbf{h}$ providing the stabilization of (2.51).

Lemma 2.2.11. *The adjoint operator of B with respect to $(.,.)_0$ is bounded, $B \in \mathcal{L}(\mathbb{H}, \mathbb{R}^2)$, and is given by*

$$B^*(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Proof. The operator B is bounded, this is then a straightforward computation. □

We define the unstable space associated to the operator $A + \delta\mathrm{I}$. Let J_u be the set of eigenvalues λ_j of A such that $\mathcal{R}e(\lambda_j) \geq -\delta$. The set J_u is exactly the set of all $\lambda \in \mathbb{C}$ such that $\lambda + \delta$

is an unstable eigenvalue of $A + \delta \mathrm{I}$. According to Lemmas 2.2.10 and 2.2.11, Hypothesis $(\boldsymbol{\mathcal{H}})_\delta$ can be rewritten as the following unique continuation property :

$$\text{For every } \lambda \in J_u \text{ and all } \phi \in \mathbb{V} \text{ that obey } (A^* - \overline{\lambda}\mathrm{I})\phi = 0 \text{ and } B^*\phi = 0, \quad \text{we have } \phi = 0, \tag{2.52}$$

where $\overline{\lambda}$ denotes the conjugate of λ.

The goal of this section is to construct a feedback control operator which uses only a finite number of scalar data to determine a control law that stabilizes problem (2.51), i.e.

Proposition 2.2.12. *Let $\delta > 0$ such that hypothesis $(\boldsymbol{\mathcal{H}})_\delta$ is fulfilled. There exists $\mathcal{K}_\delta \in \mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$ such that the operator $A + \delta \mathrm{I} + B\mathcal{K}_\delta$ generates a stable analytic semigroup on $\mathbb{H}$.*

Lemma 2.2.8 implies that the spectrum of the operator A consists of isolated eigenvalues with finite algebraic multiplicities, moreover it has no finite cluster point. In addition, the operator A generates an analytic semigroup, the control operator B belongs to $\mathcal{L}(\mathbb{R}^2, \mathbb{H})$ and we assume (2.52). Proposition 2.2.12 is then a consequence of the Fattorini criterion [10, Theorem 1.6].

In order to make the strategy clear, we provide a proof of Proposition 2.2.12.

Démonstration. We denote $\mathrm{G}(\lambda_j)$ the generalized eigenspace of A associated to the eigenvalue λ_j and we define the unstable space for $A + \delta \mathrm{I}$ by

$$\mathbb{Z}_u = \bigoplus_{\lambda \in J_u} \mathrm{G}(\lambda).$$

As $A + \delta \mathrm{I}$ generates an analytic semigroup on $\mathbb{H}$ and has compact resolvent (see Lemma 2.2.8), J_u is finite and every $\mathrm{G}(\lambda_j)$ is finite dimensional. Then the space $\mathbb{Z}_u$ is finite dimensional and we denote $d_u = \dim(\mathbb{Z}_u)$.

Besides, we can construct in the same way a space $\mathbb{Z}_s$ that contains all the stable eigenvectors of $A + \delta \mathrm{I}$, that is invariant under $(e^{tA})_{t \geq 0}$ and such that

$$\mathbb{H} = \mathbb{Z}_u \bigoplus \mathbb{Z}_s.$$

We denote Π_u the projection of $\mathbb{H}$ onto $\mathbb{Z}_u$ along $\mathbb{Z}_s$ and Π_s the projection of $\mathbb{H}$ onto $\mathbb{Z}_s$ along $\mathbb{Z}_u$. In the same way, we denote $\mathbb{Z}_u^*$ (respectively $\mathbb{Z}_s^*$) the direct sum of the generalized eigenspaces of A^* associated to an eigenvalue belonging (respectively not belonging) to J_u.

According to [66, Lemma 6.2], there exist respectively two biorthogonal families $(\mathbf{z}_i) = (\mathbf{v}_i, \theta_{1,i}, \theta_{2,i}, \omega_{1,i}, \omega_{2,i})$ and $(\tilde{\mathbf{z}}_j) = (\tilde{\mathbf{v}}_j, \tilde{\theta}_{1,j}, \tilde{\theta}_{2,j}, \tilde{\omega}_{1,j}, \tilde{\omega}_{2,j})$ of $\mathbb{Z}_u$ and $\mathbb{Z}_u^*$ satisfying the condition

$$(\mathbf{z}_i, \tilde{\mathbf{z}}_j)_0 = \delta_{ij}, \tag{2.53}$$

and $\Pi_u(\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2) = \sum_{i=1}^{d_u} \Big((\mathbf{v}, \theta_1, \theta_2, \omega_1, \omega_2), (\tilde{\mathbf{v}}_i, \tilde{\theta}_{1,i}, \tilde{\theta}_{2,i}, \tilde{\omega}_{1,i}, \tilde{\omega}_{2,i})\Big)_0 (\mathbf{v}_i, \theta_{1,i}, \theta_{2,i}, \omega_{1,i}, \omega_{2,i})$. We can also choose $\mathbf{z}_i$ and $\tilde{\mathbf{z}}_j$ to be respectively generalized eigenvectors of A and A^*.

The adjoint Π_u^* of Π_u is the projection of $\mathbb{H}$ onto $\mathbb{Z}_u^*$ along $\mathbb{Z}_s^*$ as a straightforward calculation gives

$$\forall \tilde{\mathbf{z}} \in \mathbb{H}, \ \Pi_u^* \tilde{\mathbf{z}} = \sum_{i=1}^{d_u} (\mathbf{z}_i, \tilde{\mathbf{z}})_0 \, \tilde{\mathbf{z}}_i.$$

Denoting $\mathbf{z}_u = \Pi_u \mathbf{z}$ for every $\mathbf{z} \in \mathbb{H}$, the projection of (2.51) onto $\mathbb{Z}_u$ reads

$$\begin{cases} \mathbf{z}_u'(t) = A_u \mathbf{z}_u(t) + \delta \mathbf{z}_u(t) + B_u \mathbf{h}_u(t) \qquad t > 0, \\ \mathbf{z}_u(0) = \Pi_u \mathbf{z}_0, \end{cases} \tag{2.54}$$

where $A_u = \Pi_u A$, $B_u = \Pi_u B$ and $\mathbf{h}_u = \mathbf{h}_\delta$. Note that this formulation uses $A_u((\mathrm{I} - \Pi_u)\mathbf{z}) = 0$, which holds since $\mathbb{Z}_s$ is stable under $A + \delta \mathrm{I}$.

At this point, we use the following lemma that is proven just after the end of the current proof.

Lemma 2.2.13. *Under hypothesis* $(\boldsymbol{\mathcal{H}})_\delta$, *the problem* (2.54) *is controllable on* $\mathbb{Z}_u$.

Lemma 2.2.13 implies that the problem (2.54) is stabilizable. Then the feedback control law

$$\mathbf{h}_u = -B_u^* \mathcal{R}_\delta \mathbf{z}_u,$$

stabilizes problem (2.54), where $\mathcal{R}_\delta \in \mathcal{L}(\mathbb{H})$ is the unique positive self–adjoint solution in $\mathcal{L}(\mathbb{H})$ to the Riccati equation

$$\mathcal{R}_\delta(A_u + \delta \mathrm{I}) + (A_u^* + \delta \mathrm{I})\mathcal{R}_\delta + \mathrm{I} - \mathcal{R}_\delta B_u B_u^* \mathcal{R}_\delta = 0,$$

(for more information see [54, Section 6.2]).

Now we can construct a feedback law

$$\tilde{\mathcal{K}}_\delta(\mathbf{z}_\delta) = \mathbf{h}_u = -B_u^* \mathcal{R}_\delta \Pi_u \mathbf{z}_\delta,$$

that stabilizes problem (2.51). The feedback operator $\tilde{\mathcal{K}}_\delta$ belongs to $\mathcal{L}(\mathbb{H}, \mathbb{R}^2)$ and can be extended to an element $\mathcal{K}_\delta$ of $\mathcal{L}(\mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4, \mathbb{R}^2)$,

$$\mathcal{K}_\delta \mathbf{z}_\delta = \tilde{\mathcal{K}}_\delta \Pi_{\mathbb{H}} \mathbf{z}_\delta = -B_u^* \mathcal{R}_\delta \Pi_u \Pi_{\mathbb{H}} \mathbf{z}_\delta, \text{ for all } \mathbf{z}_\delta \in \mathbf{L}^2(\mathscr{F}_s) \times \mathbb{R}^4. \tag{2.55}$$

The operator $A_u + \delta \mathrm{I} + B_u \mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{Z}_u$.

Finally, let us prove that the operator $A + \delta \mathrm{I} + B\mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{H}$. We can use the decomposition $\mathbf{z}_\delta = \Pi_u \mathbf{z}_\delta + \Pi_s \mathbf{z}_\delta$, where $\Pi_u \mathbf{z}_\delta$ and $\Pi_s \mathbf{z}_\delta$ fulfil

$$\begin{cases} \Pi_u \mathbf{z}_\delta' = (A_u + \delta \mathrm{I})\Pi_u \mathbf{z}_\delta + B_u \mathcal{K}_\delta \Pi_u \mathbf{z}_\delta \qquad t > 0, \\ \Pi_u \mathbf{z}_\delta(0) = \Pi_u \mathbf{z}_0, \end{cases}$$

$$\begin{cases} \Pi_s \mathbf{z}_\delta' = (A_s + \delta \mathrm{I})\Pi_s \mathbf{z}_\delta + B_s \mathcal{K}_\delta \Pi_u \mathbf{z}_\delta \qquad t > 0, \\ \Pi_s \mathbf{z}_\delta(0) = \Pi_s \mathbf{z}_0, \end{cases}$$

where $A_s = \Pi_s A$, $B_s = \Pi_s B$ and where we have used $\Pi_s A \Pi_u = \Pi_u A \Pi_s = 0$.

As the operator $A_u + \delta \mathrm{I} + B\mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{Z}_u$, we have $\|\Pi_u \mathbf{z}_\delta\|_{\mathrm{L}^2(0,\infty;\mathbb{H})} \leq C$. Moreover, according to the definition of J_u, the operator $A_s + \delta \mathrm{I}$ generates an analytic stable semigroup on $\mathbb{Z}_s$, then according to [24, Theorem 3.1.(i), p. 143], we have $\|\Pi_s \mathbf{z}_\delta\|_{\mathrm{L}^2(0,\infty;\mathbb{H})} \leq C$. Hence $A + \delta \mathrm{I} + B\mathcal{K}_\delta$ generates an analytic stable semigroup on $\mathbb{H}$. $\square$

Proof of Lemma 2.2.13. According to the Hautus test (see for instance [150, Proposition 1.5.1]), this result is equivalent to $\mathrm{Ker}(A_u^* + \delta \mathrm{I} - \overline{\lambda} \mathrm{I}) \cap \mathrm{Ker}(B_u^*) = \{0\}$ for every eigenvalue λ of $A_u + \delta \mathrm{I}$, i.e. $\mathrm{Ker}(A_u^* - \overline{\lambda} \mathrm{I}) \cap \mathrm{Ker}(B_u^*) = \{0\}$ for every eigenvalue λ of A_u.

The definition of $\mathbb{Z}_u$ implies that this space is stable under A. Hence, for every $\mathbf{z}_u \in \mathbb{Z}_u$, $A_u\mathbf{z}_u = A\mathbf{z}_u$. Then, the eigenvalues of A_u are the elements of J_u.

Now, we need to compute A_u^* and B_u^*, the adjoint operators of A_u and B_u with respect to $\mathbb{H}$. Let $\mathbf{z}_u^* \in \mathbb{Z}_u^*$ and $\mathbf{z} \in \mathbb{H}$, we have

$$(A_u^*\mathbf{z}_u^*,\mathbf{z})_0 = (\mathbf{z}_u^*,\Pi_u A\mathbf{z})_0 = (A^*\mathbf{z}_u^*,\mathbf{z})_0\,,$$

and

$$(B_u^*\mathbf{z}_u^*,\mathbf{z})_0 = (\mathbf{z}_u^*,\Pi_u B\mathbf{z})_0 = (B^*\mathbf{z}_u^*,\mathbf{z})_0\,,$$

then $\forall \mathbf{z}_u^* \in \mathbb{Z}_u^*$, we have $A_u^*\mathbf{z}_u^* = A^*\mathbf{z}_u^*$ and $B_u^*\mathbf{z}_u^* = B^*\mathbf{z}_u^*$.

Let us now prove the Hautus test on (A_u^*, B_u^*). Let $\lambda \in \mathbb{C}$ be an eigenvalue of A_u, then $\lambda \in J_u$. Let $\mathbf{z}_u^* \in \mathbb{Z}_u^*$ and assume that $\mathbf{z}_u^* \in \mathrm{Ker}(A_u^* - \overline{\lambda}\mathrm{I}) \cap \mathrm{Ker}(B_u^*)$. We have $(A^* - \overline{\lambda}\mathrm{I})\mathbf{z}_u^* = 0$ and $B^*\mathbf{z}_u^* = 0$. Thus, $\mathbf{z}_u^* \in D(A^*)$ and the property (2.52) implies that $\mathbf{z}_u^* = 0$ and this ends the proof. □

A consequence of Proposition 2.2.12 is the following lemma.

Lemma 2.2.14. *Let* $\mathbf{z}_0 \in \mathbb{V}$ *and* $\mathbf{F} \in \{\mathbf{f} \;\text{ with }\; e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0,\infty;\mathbb{H})\}$. *The solution* $\mathbf{z}$ *to the closed–loop problem*

$$\left\{\begin{array}{ll} \mathbf{z}' = A\mathbf{z} + B\mathcal{K}_\delta\mathbf{z} + \mathbf{F} & t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0, & \end{array}\right. \tag{2.56}$$

belongs to $\{\mathbf{z} \;\text{ with }\; e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0,\infty;D(A)) \cap \mathrm{H}^1(0,\infty;\mathbb{H}) \cap \mathscr{C}^0([0,\infty);\mathbb{V})\}$. *Moreover, we have the estimate*

$$\|e^{\delta t}\mathbf{z}\|_{\mathrm{L}^2(0,\infty;D(A))\cap \mathrm{H}^1(0,\infty;\mathbb{H})\cap\mathscr{C}^0([0,\infty);\mathbb{V})} \le C(\|\mathbf{z}_0\|_{\mathbb{V}} + \|e^{\delta t}\mathbf{F}\|_{\mathrm{L}^2(0,\infty;\mathbb{H})}). \tag{2.57}$$

Démonstration. We have $\delta\mathrm{I}+B\mathcal{K}_\delta \in \mathcal{L}(\mathbb{H})$, then $D(A+\delta\mathrm{I}+B\mathcal{K}_\delta) = D(A)$ and by interpolation for $\lambda > 0$ large enough, $D((\lambda\mathrm{I} - A - \delta\mathrm{I} - B\mathcal{K}_\delta)^{1/2}) = D((\lambda\mathrm{I} - A)^{1/2}) = \mathbb{V}$, for the definition of these spaces see (2.43)–(2.44). Moreover, $e^{\delta t}\mathbf{F} \in \mathrm{L}^2(0,\infty;\mathbb{H})$.

In addition, according to Proposition 2.2.12, $A + \delta\mathrm{I} + B\mathcal{K}_\delta$ generates an analytic semigroup that is stable on $\mathbb{H}$, hence according to [24, Theorem 3.1.(i), p. 143], there exists $\mathbf{z}_\delta \in \mathrm{H}^1(0,+\infty;\mathbb{H}) \cap \mathrm{L}^2(0,+\infty;D(A))$ satisfying

$$\left\{\begin{array}{ll} \mathbf{z}_\delta' = (A + \delta\mathrm{I} + B\mathcal{K}_\delta)\mathbf{z}_\delta + e^{\delta t}\mathbf{F} & t > 0, \\ \mathbf{z}_\delta(0) = \mathbf{z}_0. & \end{array}\right.$$

By interpolation, $\mathbf{z}_\delta$ also belongs to $\mathscr{C}^0([0,\infty);\mathbb{V})$. Now, $\mathbf{z} = e^{-\delta t}\mathbf{z}_\delta$ belongs to $\{\mathbf{z} \;\text{ with }\; e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0,\infty;D(A)) \cap \mathrm{H}^1(0,\infty;\mathbb{H}) \cap \mathscr{C}^0([0,\infty);\mathbb{V})\}$ and is solution to (2.56). Moreover, we have the estimate (2.57) as a consequence of [24, Theorem 3.1.(i), p. 143]. □

2.2.5 Stabilization of the linear system (2.30)

In this section we prove that the feedback operator $\mathcal{K}_\delta$ stabilizes the linear problem (2.30). The velocity is decomposed as $\mathbf{v} = \widetilde{\mathbf{v}} + \mathbf{v}_\mathbf{g}$ where $\mathbf{v}_\mathbf{g}$ is a lifting of $\mathbf{g}$. We first prove this result for distributed source terms only (i.e. on $\widetilde{\mathbf{v}}$) and then for boundary source terms (i.e. for $\widetilde{\mathbf{v}} + \mathbf{v}_\mathbf{g}$). The first part is a consequence of the study of the semigroup.

2.2.5.1 Stabilization of the problem with nonhomogeneous distributed source terms

In this section we prove a stabilization result for the following system that corresponds to (2.30) with $\mathbf{g} = 0$,

$$\begin{cases} \dfrac{\partial \widetilde{\mathbf{v}}}{\partial t} + (\mathbf{w}\cdot\nabla)\widetilde{\mathbf{v}} + (\widetilde{\mathbf{v}}\cdot\nabla)\mathbf{w} - \mathbf{L_F}(\widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2, \mathbf{y}) - \nu\Delta\widetilde{\mathbf{v}} + \nabla\widetilde{q} = \widetilde{\mathbf{f}} & \text{in } (0,\infty)\times\mathscr{F}_s, \\ \operatorname{div} \widetilde{\mathbf{v}} = 0 & \text{in } (0,\infty)\times\mathscr{F}_s, \\ \widetilde{\mathbf{v}} = \dot{\widetilde{\theta}}_1 \partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(0,0,.) + \dot{\widetilde{\theta}}_2 \partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(0,0,.) & \text{on } (0,\infty)\times\partial S_s, \\ \widetilde{\mathbf{v}} = 0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{D}}, \\ \sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n} = 0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{N}}, \\ \widetilde{\mathbf{v}}(0,.) = \widetilde{\mathbf{v}}_0 & \text{in } \mathscr{F}_s, \\ \mathcal{M}_{0,0}\begin{pmatrix} \ddot{\widetilde{\theta}}_1 \\ \ddot{\widetilde{\theta}}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} [\widetilde{q}\mathrm{I} - \nu(\nabla\widetilde{\mathbf{v}} + (\nabla\widetilde{\mathbf{v}})^T)]\mathbf{n}_s \cdot \partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\,\mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_s} [\widetilde{q}\mathrm{I} - \nu(\nabla\widetilde{\mathbf{v}} + (\nabla\widetilde{\mathbf{v}})^T)]\mathbf{n}_s \cdot \partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\,\mathrm{d}\gamma_y \end{pmatrix} & \\ \qquad\qquad + \mathbf{L_S}(\widetilde{\theta}_1, \widetilde{\theta}_2) + \widetilde{\mathbf{s}} + \mathbf{h} & \text{on } (0,\infty), \\ \widetilde{\theta}_1(0) = \theta_{1,0}, \quad \widetilde{\theta}_2(0) = \theta_{2,0}, & \\ \dot{\widetilde{\theta}}_1(0) = \omega_{1,0}, \quad \dot{\widetilde{\theta}}_2(0) = \omega_{2,0}. & \end{cases} \tag{2.58}$$

Proposition 2.2.15. *Let $\delta > 0$ and let $(\boldsymbol{\mathcal{H}})_\delta$ be fulfilled. For every $(\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s)\times\mathbb{R}^4$, fulfilling the compatibility conditions*

$$\begin{cases} \operatorname{div} \widetilde{\mathbf{v}}_0 = 0 & \text{in } \mathscr{F}_s, \\ \widetilde{\mathbf{v}}_0 = \displaystyle\sum_{j=1}^{2} \omega_{j,0}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,.) & \text{on } \partial S_s, \\ \widetilde{\mathbf{v}}_0 = 0 & \text{on } \Gamma_{\mathrm{D}}, \end{cases} \tag{2.59}$$

and every source terms $\widetilde{\mathbf{f}} \in \mathbb{F}^\infty_\delta$ and $\widetilde{\mathbf{s}} \in \mathbb{S}^\infty_\delta$ problem (2.58) with the control taken as $\mathbf{h} = \mathcal{K}_\delta(\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2)$ admits a unique solution $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2) \in \mathbb{U}^\infty_\delta \times \mathbb{P}^\infty_\delta \times \Theta^\infty_\delta$ with the following estimate

$$\begin{aligned} \|\widetilde{\mathbf{v}}\|_{\mathbb{U}^\infty_\delta} + \|\widetilde{q}\|_{\mathbb{P}^\infty_\delta} + \|(\widetilde{\theta}_1, \widetilde{\theta}_2)\|_{\Theta^\infty_\delta} \le C(&\|\widetilde{\mathbf{v}}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \\ &+ \|\widetilde{\mathbf{f}}\|_{\mathbb{F}^\infty_\delta} + \|\widetilde{\mathbf{s}}\|_{\mathbb{S}^\infty_\delta}), \end{aligned} \tag{2.60}$$

where C does not depend on the initial data and on the source terms.

Proof. The initial data $(\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ fulfil the compatibility conditions (2.59), that is why $\mathbf{z}_0 = (\widetilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ belongs to $\mathbb{V}$. Moreover, as $\widetilde{\mathbf{f}} \in \mathbb{F}^\infty_\delta$ and $\widetilde{\mathbf{s}} \in \mathbb{S}^\infty_\delta$,

$$\mathbf{F} = \begin{pmatrix} \widetilde{\mathbf{f}} \\ 0 \\ 0 \\ \mathcal{M}_{0,0}^{-1}\widetilde{\mathbf{s}} \end{pmatrix},$$

belongs to $\{\mathbf{f}\ \text{ with }\ e^{\delta t}\mathbf{f} \in \mathrm{L}^2(0,\infty;\mathbb{H})\}$. Then, according to Lemma 2.2.14, the solution to (2.56) belongs to $\{\mathbf{z}\ \text{ with }\ e^{\delta t}\mathbf{z} \in \mathrm{L}^2(0,\infty;D(A)) \cap \mathrm{H}^1(0,\infty;\mathbb{H}) \cap \mathscr{C}^0([0,\infty);\mathbb{V})\}$.

Now, we use the identity $\mathbf{z} = (\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2)$. According to Lemmas 2.2.2 and 2.2.6, $\widetilde{\mathbf{v}} \in \mathbb{U}_\delta^\infty$, $(\widetilde{\theta}_1, \widetilde{\theta}_2) \in \Theta_\delta^\infty$ and there exists $\widetilde{q} \in \mathbb{P}_\delta^\infty$ such that problem (2.56) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\widetilde{\mathbf{v}}\\ \widetilde{\theta}_1\\ \widetilde{\theta}_2\\ \widetilde{\omega}_1\\ \widetilde{\omega}_2\end{pmatrix} = \begin{pmatrix}\operatorname{div}\sigma_F(\widetilde{\mathbf{v}}, \widetilde{q}) + \mathbf{L}_\mathbf{F}(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2, \mathbf{y}) - (\widetilde{\mathbf{v}}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\widetilde{\mathbf{v}} + \widetilde{\mathbf{f}}\\ \widetilde{\omega}_1\\ \widetilde{\omega}_2\\ \mathcal{M}_{0,0}^{-1}\left(\begin{pmatrix}\displaystyle\int_{\partial S_s} -\sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi}^\mathbf{S}(0,0,\gamma_y)\\ \displaystyle\int_{\partial S_s} -\sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi}^\mathbf{S}(0,0,\gamma_y)\end{pmatrix} + \mathbf{L}_\mathbf{S}(\widetilde{\theta}_1, \widetilde{\theta}_2) + \mathcal{K}_\delta(\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2) + \widetilde{\mathbf{s}}\right)\end{pmatrix},$$

where $\mathbf{L}_\mathbf{F}$ and $\mathbf{L}_\mathbf{S}$ are defined in (2.31)–(2.32). Then, $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2)$ is solution to (2.58).

Finally, the estimate (2.60) is a consequence of (2.57). □

2.2.5.2 A first stabilization result for the problem with a nonhomogeneous boundary datum

In the sequel, for $\mathbf{g} \in \mathbb{G}_\delta^\infty$ we consider $\mathbf{v}_\mathbf{g} \in \{\mathbf{v} \ \text{ with } \ e^{\delta t}\mathbf{v} \in \mathrm{H}^1(0,\infty;\mathbf{H}^2(\mathscr{F}_s))\}$ that satisfies

$$\begin{cases}\mathbf{v}_\mathbf{g} = \mathbf{g} & \text{on } (0,\infty)\times\partial S_s,\\ \operatorname{div}\mathbf{v}_\mathbf{g} = 0 & \text{in } (0,\infty)\times\mathscr{F}_s,\\ \mathbf{v}_\mathbf{g} = 0 & \text{on } (0,\infty)\times\Gamma_\mathrm{D},\\ (\nabla\mathbf{v}_\mathbf{g} + (\nabla\mathbf{v}_\mathbf{g})^T)\mathbf{n}_s = 0 & \text{on } (0,\infty)\times\Gamma_\mathrm{N},\end{cases} \tag{2.61}$$

and

$$\|\mathbf{v}_\mathbf{g}e^{\delta t}\|_{\mathrm{H}^1(0,\infty;\mathbf{H}^2(\mathscr{F}_s))} \leq C\|\mathbf{g}\|_{\mathbb{G}_\delta^\infty}, \tag{2.62}$$

see [118, Theorem 2.16] for a proof of the existence of $\mathbf{v}_\mathbf{g}$.

The following proposition enables a stabilization of the problem (2.30) with $\mathbf{g} \neq 0$. However, contrary to Proposition 2.2.1, the feedback control is $\mathcal{K}_\delta(\mathbf{v} - \mathbf{v}_\mathbf{g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ instead of $\mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$.

Proposition 2.2.16. *Let $\delta > 0$ and let $(\boldsymbol{\mathcal{H}})_\delta$ be fulfilled. For every $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s)\times\mathbb{R}^4$, $\mathbf{f} \in \mathbb{F}_\delta^\infty$, $\mathbf{g} \in \mathbb{G}_\delta^\infty$ and $\mathbf{s} \in \mathbb{S}_\delta^\infty$ fulfilling the compatibility conditions* (2.39)*, problem* (2.30) *with the control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v}_\mathbf{g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ admits a unique solution $(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ with the following estimate,*

$$\begin{aligned}\|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1,\theta_2)\|_{\Theta_\delta^\infty} \leq C(&\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}|\\ &+ \|\mathbf{f}\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}\|_{\mathbb{S}_\delta^\infty}),\end{aligned} \tag{2.63}$$

where C does not depend on the initial conditions and the source terms.

Proof. Let $(\mathbf{v}, q, \theta_1, \theta_2)$ be the solution to (2.30) with $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v}_\mathbf{g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$. We now consider $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2) = (\mathbf{v} - \mathbf{v}_\mathbf{g}, q, \theta_1, \theta_2)$, it is solution to problem (2.58) where

$$\begin{aligned}&\widetilde{\mathbf{v}}_0 = \mathbf{v}_0 - \mathbf{v}_\mathbf{g}(0),\\ &\widetilde{\mathbf{f}} = \mathbf{f} - \partial_t\mathbf{v}_\mathbf{g} - (\mathbf{w}\cdot\nabla)\mathbf{v}_\mathbf{g} - (\mathbf{v}_\mathbf{g}\cdot\nabla)\mathbf{w} + \nu\Delta\mathbf{v}_\mathbf{g},\\ &\widetilde{\mathbf{s}} = \mathbf{s} - \left(\int_{\partial S_s}\nu(\nabla\mathbf{v}_\mathbf{g} + \nabla\mathbf{v}_\mathbf{g}^T)\mathbf{n}_s\cdot\partial_{\theta_j}\mathbf{\Phi}^\mathbf{S}(0,0,\gamma_y)\right)_{j=1,2}.\end{aligned}$$

The initial data $(\tilde{\mathbf{v}}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ fulfil (2.59), then all the terms have the good regularity and we have the estimate (2.60). We now use (2.62) and get

$$\begin{aligned} \|\widetilde{\mathbf{f}}\|_{\mathbb{F}^\infty_\delta} &\leq \|\mathbf{f}\|_{\mathbb{F}^\infty_\delta} + C\|\mathbf{g}\|_{\mathbb{G}^\infty_\delta}, \\ \|\widetilde{\mathbf{s}}\|_{\mathbb{S}^\infty_\delta} &\leq \|\mathbf{s}\|_{\mathbb{S}^\infty_\delta} + C\|\mathbf{g}\|_{\mathbb{G}^\infty_\delta}. \end{aligned}$$

All these estimates prove the estimate (2.63). □

2.2.5.3 Proof of Proposition 2.2.1

Proof. In Proposition 2.2.16, we have proven that the control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ stabilizes the problem (2.30). We now want to prove that the control $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ also stabilizes the same problem.

Let $(\mathbf{v}, q, \theta_1, \theta_2)$ be the solution to (2.30) with $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v} - \mathbf{v_g}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$. According to Proposition 2.2.16, we have the estimate (2.63).

Let $(\widehat{\mathbf{v}}, \widehat{q}, \widehat{\theta}_1, \widehat{\theta}_2)$ be the solution to (2.30) with $\mathbf{h} = \mathcal{K}_\delta(\widehat{\mathbf{v}}, \widehat{\theta}_1, \widehat{\theta}_2, \dot{\widehat{\theta}}_1, \dot{\widehat{\theta}}_2)$. We now consider

$$(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\theta}_1, \widetilde{\theta}_2) = (\mathbf{v}, q, \theta_1, \theta_2) - (\widehat{\mathbf{v}}, \widehat{q}, \widehat{\theta}_1, \widehat{\theta}_2),$$

it is solution to (2.58) with $\mathbf{f} = 0$, $\mathbf{v}_0 = 0$, $\mathbf{s} = -\mathcal{K}_\delta(\mathbf{v_g}, 0, 0, 0, 0)$, $\theta_{1,0} = 0$, $\theta_{2,0} = 0$, $\omega_{1,0} = 0$, $\omega_{2,0} = 0$ and $\mathbf{h} = \mathcal{K}_\delta(\widetilde{\mathbf{v}}, \widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2)$. The initial data $(0, 0, 0, 0, 0)$ fulfil the compatibility conditions (2.59), hence we have the estimate (2.60). When combining the estimates (2.60) and (2.63), we get

$$\begin{aligned} \|\widehat{\mathbf{v}}\|_{\mathbb{U}^\infty_\delta} + \|\widehat{q}\|_{\mathbb{P}^\infty_\delta} + \|(\widehat{\theta}_1, \widehat{\theta}_2)\|_{\Theta^\infty_\delta} \leq{}& \|\mathbf{v}\|_{\mathbb{U}^\infty_\delta} + \|\widetilde{\mathbf{v}}\|_{\mathbb{U}^\infty_\delta} + \|q\|_{\mathbb{P}^\infty_\delta} + \|\widetilde{q}\|_{\mathbb{P}^\infty_\delta} + \|(\theta_1, \theta_2)\|_{\Theta^\infty_\delta} + \|(\widetilde{\theta}_1, \widetilde{\theta}_2)\|_{\Theta^\infty_\delta} \\ \leq{}& C(\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}\|_{\mathbb{F}^\infty_\delta} + \|\mathbf{g}\|_{\mathbb{G}^\infty_\delta} + \|\mathbf{s}\|_{\mathbb{S}^\infty_\delta}). \end{aligned}$$

This ends the proof of Proposition 2.2.1. □

2.3 Stabilization of the nonlinear closed loop system

The proof of Theorem 2.1.5 will be developed in this section. As in Section 2.2 we consider $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f_s}) \in \mathbf{W}^{1,\infty}(\Omega) \times \mathrm{U}^i \times \mathbb{R}^2$ and a stationary state $(\mathbf{w}, p_\mathbf{w}) \in \mathbf{H}^2_\beta(\mathscr{F}_s) \times \mathrm{H}^1_\beta(\mathscr{F}_s)$ that fulfil (2.16) (see Remark 2.1.8).

2.3.1 The nonlinear problem in a fixed domain

In this section, we are interested in writing the equations fulfilled by the difference between the solution to (2.14) and the stationary state. In order to do so, we consider the change of variables

$$\forall \mathbf{y} \in \mathscr{F}_s,\ \forall t \in (0, \infty), \quad \begin{cases} \mathbf{v}(t, \mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1(t), \theta_2(t), \mathbf{y}))^T \mathbf{u}(t, \mathbf{\Phi^S}(\theta_1(t), \theta_2(t), \mathbf{y})) - \mathbf{w}(\mathbf{y}), \\ q(t, \mathbf{y}) = p(t, \mathbf{\Phi^S}(\theta_1(t), \theta_2(t), \mathbf{y})) - p_\mathbf{w}(\mathbf{y}), \end{cases} \tag{2.64}$$

where $\mathbf{\Phi^S}$ is the diffeomorphism defined in (2.18) fulfilling (2.20), moreover $\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1, \theta_2, \mathbf{y}) = \nabla_\mathbf{y} \mathbf{\Phi^S}(\theta_1, \theta_2, \mathbf{y})$ and $\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^S}})$ is the cofactor matrix of $\mathcal{J}_{\mathbf{\Phi^S}}$. This change of variables has been chosen in order to have a divergence free velocity $\mathbf{v}$ in the fixed domain.

One can show that under the feedback control $\mathbf{h}=\mathcal{K}_\delta(\mathbf{v},\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)$ defined in Proposition 2.2.1, $(\mathbf{v},q,\theta_1,\theta_2)$ is solution to the closed loop system

$$\begin{cases}\dfrac{\partial \mathbf{v}}{\partial t}+(\mathbf{w}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{w}-\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y})-\nu\Delta\mathbf{v}+\nabla q=\mathbf{f} & \text{in } (0,\infty)\times\mathscr{F}_s,\\ \operatorname{div}\mathbf{v}=0 & \text{in } (0,\infty)\times\mathscr{F}_s,\\ \mathbf{v}=\dot\theta_1\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)+\dot\theta_2\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)+\mathbf{g} & \text{on } (0,\infty)\times\partial S_s,\\ \mathbf{v}=0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{D}},\\ \sigma_F(\mathbf{v},q)\mathbf{n}=0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{N}},\\ \mathbf{v}(0,\mathbf{y})=\mathbf{v}_0(\mathbf{y})=\operatorname{cof}(\mathcal{J}_{\mathbf{\Phi^S}}(\theta_{1,0},\theta_{2,0},\mathbf{y}))^T\mathbf{u}_0(\mathbf{\Phi^S}(\theta_{1,0},\theta_{2,0},\mathbf{y}))-\mathbf{w}(\mathbf{y}) & \text{in } \mathscr{F}_s,\\ \mathcal{M}_{0,0}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix}=\begin{pmatrix}\displaystyle\int_{\partial S_s}-\sigma_F(\mathbf{v},q)\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)\\ \displaystyle\int_{\partial S_s}-\sigma_F(\mathbf{v},q)\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)\end{pmatrix} & \\ \qquad\qquad+\mathbf{L_S}(\theta_1,\theta_2)+\mathbf{s}+\mathcal{K}_\delta(\mathbf{v},\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) & \text{on } (0,\infty),\\ \theta_1(0)=\theta_{1,0},\quad \theta_2(0)=\theta_{2,0}, & \\ \dot\theta_1(0)=\omega_{1,0},\quad \dot\theta_2(0)=\omega_{2,0}, & \end{cases}\tag{2.65}$$

where the linear terms are

$$\begin{aligned}\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y})&=\mathbf{L}_1(\mathbf{y})\theta_1+\mathbf{L}_2(\mathbf{y})\theta_2+\mathbf{L}_3(\mathbf{y})\dot\theta_1+\mathbf{L}_4(\mathbf{y})\dot\theta_2,\quad\forall\mathbf{y}\in\mathscr{F}_s,\\ \mathbf{L_S}(\theta_1,\theta_2)&=\mathbf{L}_5\theta_1+\mathbf{L}_6\theta_2,\end{aligned}$$

and the coefficients $\mathbf{L}_1$–$\mathbf{L}_6$ are defined in Appendix B. Moreover $\mathcal{K}_\delta$ is the feedback operator given in (2.55) and the source terms are given by the nonlinear (at least quadratic) terms

$$\begin{cases}\mathbf{f}=\mathbf{f}^{NL}(\theta_1,\theta_2,\mathbf{v},q),\\ \mathbf{g}=\mathbf{g}^{NL}(\theta_1,\theta_2),\\ \mathbf{s}=\mathbf{s}^{NL}(\theta_1,\theta_2,\mathbf{v},q),\end{cases}\tag{2.66}$$

defined below

$$\begin{cases}\mathbf{f}^{NL}(\theta_1,\theta_2,\mathbf{v},q)=\mathbf{F}(\theta_1,\theta_2,\mathbf{w}+\mathbf{v},p_{\mathbf{w}}+q)+(\mathbf{w}\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{w}\\ \qquad\qquad -\mathbf{L_F}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,.)+\mathbf{f}_{\mathscr{F}}(\mathbf{\Phi^S}(\theta_1,\theta_2,.))-\mathbf{f}_{\mathscr{F}},\\ \mathbf{g}^{NL}(\theta_1,\theta_2)=\mathbf{G}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2),\\ \mathbf{s}^{NL}(\theta_1,\theta_2,\mathbf{v},q)=\mathbf{S}(\theta_1,\theta_2,\mathbf{w}+\mathbf{v},p_{\mathbf{w}}+q)-\mathbf{L_S}(\theta_1,\theta_2),\end{cases}\tag{2.67}$$

where $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{S}$ are defined as follows

$$\begin{aligned}&\mathbf{F}(\theta_1,\theta_2,\mathbf{v},q)=\mathbf{F}^1(\theta_1,\theta_2,\mathbf{v})+\mathbf{F}^2(\theta_1,\theta_2,\mathbf{v})+\mathbf{F}^3(\theta_1,\theta_2,\mathbf{v})+\mathbf{F}^4(\theta_1,\theta_2,\mathbf{v})+\mathbf{F}^5(\theta_1,\theta_2,q),\\ &\mathbf{F}^1(\theta_1,\theta_2,\mathbf{v})=(\mathrm{I}-\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1,\theta_2,\mathbf{\Phi^S}))^T)\frac{\partial\mathbf{v}}{\partial t},\\ &\mathbf{F}^2(\theta_1,\theta_2,\mathbf{v})=-\operatorname{cof}(\dot\theta_1\nabla_{\mathbf{x}}\partial_{\theta_1}\mathbf{\Psi^S}(\theta_1,\theta_2,\mathbf{\Phi^S})+\dot\theta_2\nabla_{\mathbf{x}}\partial_{\theta_2}\mathbf{\Psi^S}(\theta_1,\theta_2,\mathbf{\Phi^S}))^T\mathbf{v}\\ &\qquad -\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1,\theta_2,\mathbf{\Phi^S}))^T(\nabla_{\mathbf{y}}\mathbf{v})\Big(\dot\theta_1\partial_{\theta_1}\mathbf{\Psi^S}(\theta_1,\theta_2,\mathbf{\Phi^S})+\dot\theta_2\partial_{\theta_2}\mathbf{\Psi^S}(\theta_1,\theta_2,\mathbf{\Phi^S})\Big),\\ &\mathbf{F}^3(\theta_1,\theta_2,\mathbf{v})_i=\nu\sum_{j,k,\ell,m}\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1,\theta_2,\mathbf{\Phi^S}))_{ki}\frac{\partial^2 v_k}{\partial y_\ell\partial y_m}\frac{\partial\Psi^S_\ell}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi^S})\frac{\partial\Psi^S_m}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi^S})\\ &\qquad+2\nu\sum_{j,k,\ell}\operatorname{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1,\theta_2,\mathbf{\Phi^S}))_{ki}\frac{\partial v_k}{\partial y_\ell}\frac{\partial\Psi^S_\ell}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi^S})\\ &\qquad+\nu\sum_{j,k,\ell}\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1,\theta_2,\mathbf{\Phi^S}))_{ki}\frac{\partial v_k}{\partial y_\ell}\frac{\partial^2\Psi^S_\ell}{\partial x_j^2}(\theta_1,\theta_2,\mathbf{\Phi^S})\\ &\qquad+\nu\sum_{j,k}\operatorname{cof}(\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1,\theta_2,\mathbf{\Phi^S}))_{ki}v_k-\nu\Delta_{\mathbf{y}}v_i(t,\mathbf{y}),\end{aligned}\tag{2.68}$$

$$\mathbf{F}^4(\theta_1,\theta_2,\mathbf{v})_i = -\sum_{j,k,r} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}))_{kj} \mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}))_{ri} v_k v_r$$
$$- \sum_{k,r} \det(\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}))^2 \frac{\partial \Phi_i^S}{\partial y_r}\frac{\partial v_r}{\partial y_k} v_k,$$
$$\mathbf{F}^5(\theta_1,\theta_2,q) = (\mathrm{I} - \mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}))^T \nabla_{\mathbf{y}} q,$$

$$\begin{aligned}
&\mathbf{G}(\theta_1,\theta_2,\omega_1,\omega_2) = \sum_{j=1}^2 \omega_j \Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1,\theta_2))^T \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,\mathbf{y}) - \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}) \Big),\\
&\mathbf{S}(\theta_1,\theta_2,\mathbf{v},q) = -(\mathcal{M}_{\theta_1,\theta_2} - \mathcal{M}_{0,0}) \begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} + \mathbf{M}_{\mathbf{I}}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)\\
&+ \begin{pmatrix} \int_{\partial S_s} |\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1,\theta_2,\gamma_y)\mathbf{t}_s|\ [q\mathrm{I} - \nu(\mathcal{G}(\theta_1,\theta_2,\mathbf{v}) + \mathcal{G}(\theta_1,\theta_2,\mathbf{v})^T)]\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi}^{\mathbf{S}}) \cdot \partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,\gamma_y) \\ \int_{\partial S_s} |\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1,\theta_2,\gamma_y)\mathbf{t}_s|\ [q\mathrm{I} - \nu(\mathcal{G}(\theta_1,\theta_2,\mathbf{v}) + \mathcal{G}(\theta_1,\theta_2,\mathbf{v})^T)]\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi}^{\mathbf{S}}) \cdot \partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,\gamma_y) \end{pmatrix}\\
&- \begin{pmatrix} \int_{\partial S_s} [q\mathrm{I} - \nu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)]\mathbf{n}_s \cdot \partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \\ \int_{\partial S_s} [q\mathrm{I} - \nu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)]\mathbf{n}_s \cdot \partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \end{pmatrix},
\end{aligned} \tag{2.69}$$

where $\mathbf{M}_{\mathbf{I}}$ and $\mathcal{M}_{\theta_1,\theta_2}$ are defined in (2.10), (2.11) and

$$\begin{aligned}
\mathcal{G}(\theta_1,\theta_2,\mathbf{v})_{ij} = \sum_k \mathrm{cof}\left[\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,.)) \circ \mathbf{\Phi}^{\mathbf{S}}\right]_{ki} v_k \\
+ \sum_{k,\ell} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}))_{ki} \frac{\partial v_k}{\partial y_\ell}\frac{\partial \Psi_\ell^S}{\partial x_j}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}).
\end{aligned} \tag{2.70}$$

For the sake of intelligibility, we have used the notation $\mathbf{\Phi}^{\mathbf{S}} = \mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,.)$.

2.3.2 Proof of the stabilization result in the fixed domain

In this section, we develop the fixed point argument used to prove the stabilization result of the nonlinear problem (2.14) in the fixed domain $\mathscr{F}_s$ (i.e. the stabilization result for (2.65)–(2.66)).

Proposition 2.3.1. *Let $\delta > 0$ and let $(\boldsymbol{\mathcal{H}})_\delta$ be fulfilled. There exists $\varepsilon > 0$, such that for every $(\mathbf{v}_0,\theta_{1,0},\theta_{2,0},\omega_{1,0},\omega_{2,0}) \in \mathbf{H}^1(\mathscr{F}_s) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ satisfying the compatibility conditions*

$$\begin{cases} \mathrm{div}\ \mathbf{v}_0 = 0 & \text{in } \mathscr{F}_s,\\ \mathbf{v}_0 = \sum_{j=1}^2 \omega_{j,0}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_{1,0},\theta_{2,0},.))^T \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(\theta_{1,0},\theta_{2,0},.) & \text{on } \partial S_s,\\ \mathbf{v}_0 = 0 & \text{on } \Gamma_{\mathrm{D}}, \end{cases} \tag{2.71}$$

and

$$\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \le \varepsilon, \tag{2.72}$$

problem (2.65)–(2.66) *admits a solution* $(\mathbf{v},q,\theta_1,\theta_2)$ *that tends to zero with exponential decay rate δ. For all $t > 0$, we have*

$$\|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\omega_1(t)| + |\omega_2(t)| \le Ce^{-\delta t}. \tag{2.73}$$

Proof. Let $(\mathbf{v}_0, \theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ in $\mathbf{H}^1(\mathscr{F}_s) \times \mathbb{D}_\Theta \times \mathbb{R}^2$ fulfilling the compatibility conditions (2.71) and (2.72) for some $\varepsilon > 0$. We consider the space

$$\mathbb{N}_\delta^\infty = \left\{ \begin{array}{r} (\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty \text{ with } (\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)(0) = (\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0}), \\ \text{for every } t \text{ in } (0,\infty),\ (\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta \end{array} \right\}, \quad (2.74)$$

equipped with the natural norm of $\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$

$$\|(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} = \|\mathbf{v}\|_{\mathbb{U}_\delta^\infty} + \|q\|_{\mathbb{P}_\delta^\infty} + \|(\theta_1, \theta_2)\|_{\Theta_\delta^\infty},$$

where all the spaces used are defined in (2.33)–(2.38).

We define the application Λ^∞ on $\mathbb{N}_\delta^\infty$ as follows. For $(\overline{\mathbf{v}}, \overline{q}, \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{N}_\delta^\infty$, $(\mathbf{v}, q, \theta_1, \theta_2) = \Lambda^\infty(\overline{\mathbf{v}}, \overline{q}, \overline{\theta}_1, \overline{\theta}_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty$ is the solution to problem (2.65) where the source terms are $\mathbf{f} = \mathbf{f}^{NL}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{v}}, \overline{q})$, $\mathbf{g} = \mathbf{g}^{NL}(\overline{\theta}_1, \overline{\theta}_2)$ and $\mathbf{s} = \mathbf{s}^{NL}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{v}}, \overline{q})$ (defined in (2.67)). Note that Λ^∞ depends on the initial conditions.

The boundary conditions (2.71) correspond to (2.39) with $\mathbf{g}(0) = \mathbf{G}(\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$ and the conditions in (2.74) impose that $\mathbf{g}^{NL}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{v}}, \overline{q})(0) = \mathbf{G}(\theta_{1,0}, \theta_{2,0}, \omega_{1,0}, \omega_{2,0})$. Moreover, we prove later in (2.75) that, for every $(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{v}}, \overline{q}) \in \mathbb{N}_\delta^\infty$, we have $\mathbf{f}^{NL}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{v}}, \overline{q}) \in \mathbb{F}_\delta^\infty$, $\mathbf{g}^{NL}(\overline{\theta}_1, \overline{\theta}_2) \in \mathbb{G}_\delta^\infty$ and $\mathbf{s}^{NL}(\overline{\theta}_1, \overline{\theta}_2, \overline{\mathbf{v}}, \overline{q}) \in \mathbb{S}_\delta^\infty$. Hence, according to Proposition 2.2.1, Λ^∞ is well defined.

The domain $\mathbb{D}_\Theta$ is open and $(0,0) \in \mathbb{D}_\Theta$, then there exists $R_0 > 0$ such that $B((0,0), R_0) \subset \mathbb{D}_\Theta$. The application Λ^∞ will be studied on a ball of radius $R \leq R_0$

$$\mathbb{B}_R^\infty = \{(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{N}_\delta^\infty \text{ with } \|(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq R\}.$$

We also consider the space

$$\widetilde{\mathbb{B}_R^\infty} = \left\{ \begin{array}{r} (\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty \text{ with } \|(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq R, \\ \forall t \in (0,\infty),\ (\theta_1, \theta_2)(t) \in \mathbb{D}_\Theta \end{array} \right\},$$

in which the initial value for $(\theta_1, \theta_2, \dot\theta_1, \dot\theta_2)$ is free. The following lemma will be used.

Lemma 2.3.2. *There exists a constant $C' = C'(R_0)$, such that for every $R \in (0, R_0)$, every $(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a)$ and $(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b) \in \widetilde{\mathbb{B}_R^\infty}$, the following estimates hold*

$$\|\mathbf{f}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{f}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)\|_{\mathbb{F}_\delta^\infty} \leq C'R(\|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a - q^b\|_{\mathbb{P}_\delta^\infty} + \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}),$$

$$\|\mathbf{g}^{NL}(\theta_1^a, \theta_2^a) - \mathbf{g}^{NL}(\theta_1^b, \theta_2^b)\|_{\mathbb{G}_\delta^\infty} \leq C'R\|\theta^a - \theta^b\|_{\Theta_\delta^\infty},$$

$$\|\mathbf{s}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{s}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)\|_{\mathbb{S}_\delta^\infty} \leq C'R(\|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a - q^b\|_{\mathbb{P}_\delta^\infty} + \|\theta^a - \theta^b\|_{\Theta_\delta^\infty}).$$

This lemma is proven in Appendix D.2. We use the spaces $\mathbf{H}^2_\beta(\mathscr{F}_s)$ and $\mathrm{H}^1_\beta(\mathscr{F}_s)$ in this proof.

We now denote C the constant in (2.40) and C' the one in Lemma 2.3.2.

At this point, we use Lemma 2.3.2 with $(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b) = (0,0,0,0)$, we thus get for every $R > 0$ and for every $(\mathbf{v}, q, \theta_1, \theta_2)$ in $\mathbb{B}_R^\infty$,

$$\|\mathbf{f}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{F}_\delta^\infty} + \|\mathbf{g}^{NL}(\theta_1, \theta_2)\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{S}_\delta^\infty} \leq 3C'R^2. \quad (2.75)$$

Let $R = \min(R_0, 1/(6CC'))$ and $\varepsilon = R/(2C)$, hence $3CC'R^2 \leq R/2$.

We can build (θ_1, θ_2) as the solution to

$$\begin{cases} \ddot{\theta}_j(t) + 2\delta\dot{\theta}_j(t) + 2\delta^2\theta_j(t) = 0, & t > 0, \\ \theta_j(0) = \theta_{j,0}, \qquad \dot{\theta}_j(0) = \omega_{j,0}. \end{cases}$$

For ε small enough and $|\omega_{j,0}| + |\theta_{j,0}| \leq \varepsilon$, $(0, 0, \theta_1, \theta_2)$ belongs to $\mathbb{B}_R^\infty$. Thus, taking $\varepsilon > 0$ smaller if necessary, we can assume that $\mathbb{B}_R^\infty \neq \emptyset$. In the sequel, we choose such a ε.

According to Proposition 2.2.1, we have for every $(\mathbf{v}, q, \theta_1, \theta_2)$ in $\mathbb{B}_R^\infty$,

$$\begin{aligned} \|\Lambda^\infty(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq C(\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| + \|\mathbf{f}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{F}_\delta^\infty} \\ + \|\mathbf{g}^{NL}(\theta_1, \theta_2)\|_{\mathbb{G}_\delta^\infty} + \|\mathbf{s}^{NL}(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{S}_\delta^\infty}), \end{aligned}$$

we combine this result with (2.75). Then, by using (2.72), $\varepsilon = R/(2C)$ and $3CC'R^2 \leq R/2$, we have

$$\|\Lambda^\infty(\mathbf{v}, q, \theta_1, \theta_2)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \leq R.$$

Hence, if we write $(\tilde{\mathbf{v}}, \tilde{q}, \tilde{\theta}_1, \tilde{\theta}_2) = \Lambda^\infty(\theta_1, \theta_2, \mathbf{v}, q)$, then we have

$$\|(\tilde{\theta}_1, \tilde{\theta}_2)\|_{\mathbf{L}^\infty(0,\infty)} \leq \|\Lambda^\infty(\theta_1, \theta_2, \mathbf{v}, q)\|_{\mathbb{N}_\delta^\infty} \leq R \leq R_0.$$

Then, $(\tilde{\theta}_1, \tilde{\theta}_2)$ belongs to $\mathbb{D}_\Theta$. This proves that $\Lambda^\infty : \mathbb{B}_R^\infty \to \mathbb{B}_R^\infty$.

For $(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a)$ and $(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b)$ in $\mathbb{B}_R^\infty$, $\Lambda^\infty(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a) - \Lambda^\infty(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b)$ solves problem (2.65) where the source terms are $\mathbf{f}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{f}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)$, $\mathbf{g}^{NL}(\theta_1^a, \theta_2^a) - \mathbf{g}^{NL}(\theta_1^b, \theta_2^b)$, $\mathbf{s}^{NL}(\theta_1^a, \theta_2^a, \mathbf{v}^a, q^a) - \mathbf{s}^{NL}(\theta_1^b, \theta_2^b, \mathbf{v}^b, q^b)$ and the initial data are null. Then, according to Proposition 2.2.1 and Lemma 2.3.2, we have

$$\begin{aligned} \|\Lambda^\infty(\mathbf{v}^a, q^a, \theta_1^a, \theta_2^a) - \Lambda^\infty(\mathbf{v}^b, q^b, \theta_1^b, \theta_2^b)\|_{\mathbb{U}_\delta^\infty \times \mathbb{P}_\delta^\infty \times \Theta_\delta^\infty} \\ \leq 3CC'R(\|\theta^a - \theta^b\|_{\Theta_\delta^\infty} + \|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_\delta^\infty} + \|q^a - q^b\|_{\mathbb{P}_\delta^\infty}). \end{aligned} \tag{2.76}$$

As $R \leq 1/(6CC')$, the estimate (2.76) yields that Λ^∞ is a contraction on $\mathbb{B}_R^\infty$. Hence, according to the Picard fixed point theorem, there exists a unique fixed point $(\mathbf{v}, q, \theta_1, \theta_2)$ to Λ^∞ in $\mathbb{B}_R^\infty$. This fixed point solves the closed–loop nonlinear problem (2.65)–(2.66).

Estimate (2.73) is then a consequence of the fact that $(\mathbf{v}, q, \theta_1, \theta_2) \in \mathbb{B}_R^\infty$:

$$\|\mathbf{v}e^{\delta t}\|_{\mathscr{C}^0([0,\infty);\mathbf{H}^1(\mathscr{F}_s))} + \|\theta e^{\delta t}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{D}_\Theta)} + \|\dot{\theta} e^{\delta t}\|_{\mathrm{L}^\infty(0,\infty;\mathbb{R}^2)} \leq R \leq R_0.$$

□

2.3.3 Proof of Theorem 2.1.5

Proof. The last step towards proving Theorem 2.1.5 is to use the change of variables (2.64) to prove the result on $(\mathbf{u}, p)$ as a consequence of Proposition 2.3.1 that states properties of $(\mathbf{v}, q)$. The only difficulty is to handle the nonlinear term $\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}}\mathbf{s})^T$ that is present in this change of variable. This nonlinearity creates some difficulties for using the smallness assumption and compatibility conditions on the initial data.

Let $\delta > 0$, we consider that $(\boldsymbol{\mathcal{H}})_\delta$ is fulfilled and that the initial data satisfy the compatibility conditions (2.23). Moreover, we consider that for some $\varepsilon_1 > 0$ we have

$$\|\mathbf{u}_0(\mathbf{\Phi}^{\mathbf{S}}(\theta_{1,0}, \theta_{2,0}, .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_{1,0}| + |\theta_{2,0}| + |\omega_{1,0}| + |\omega_{2,0}| \leq \varepsilon_1.$$

Since $\mathbf{v}_0(\mathbf{y}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}^\mathbf{S}}(\theta_{1,0}, \theta_{2,0}, \mathbf{y}))^T \mathbf{u}_0 \circ \mathbf{\Phi}^\mathbf{S}(\theta_{1,0}, \theta_{2,0}, \mathbf{y}) - \mathbf{w}(\mathbf{y})$, we have (2.71) and

$$\begin{aligned}\|\mathbf{v}_0\|_{\mathbf{H}^1(\mathscr{F}_s)} &\leq \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}^\mathbf{S}}(\theta_{1,0}, \theta_{2,0}, \mathbf{\Phi}^\mathbf{S}))^T(\mathbf{u}_0 \circ \mathbf{\Phi}^\mathbf{S} - \mathbf{w})\|_{\mathbf{H}^1(\mathscr{F}_s)} \\ &\qquad + \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}^\mathbf{S}}(\theta_{1,0}, \theta_{2,0}, \mathbf{\Phi}^\mathbf{S}) - \mathrm{I})^T \mathbf{w}\|_{\mathbf{H}^1(\mathscr{F}_s)} \\ &\leq K\varepsilon_1 + K\|\mathbf{w}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)}\varepsilon_1,\end{aligned}$$

where K is the constant in (D.18). In order to fulfil (2.72), we choose

$$\varepsilon_1 = \frac{\varepsilon}{K + K\|\mathbf{w}\|_{\mathbf{H}^2_\beta(\mathscr{F}_s)}},$$

where $\varepsilon > 0$ is the bound in Proposition 2.3.1, we have (2.72). Then, according to Proposition 2.3.1, the feedback $\mathbf{h} = \mathcal{K}_\delta(\mathbf{v}, \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ stabilizes problem (2.65)–(2.66) at decay rate δ. We denote $(\mathbf{v}, q, \theta_1, \theta_2)$ the solution to (2.65)–(2.66).

We use the identity

$$\begin{aligned}&\forall t \in (0, \infty),\ \forall \mathbf{x} \in \mathscr{F}(\theta_1(t), \theta_2(t)), \\ &\qquad \begin{cases} \mathbf{u}(t, \mathbf{x}) = \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}^\mathbf{S}}(\theta_1(t), \theta_2(t), \mathbf{x}))^T \Big(\mathbf{v}(t, \mathbf{\Psi}^\mathbf{S}(\theta_1(t), \theta_2(t), \mathbf{x})) + \mathbf{w}(\mathbf{\Psi}^\mathbf{S}(\theta_1(t), \theta_2(t), \mathbf{x}))\Big), \\ p(t, \mathbf{x}) = q(t, \mathbf{\Psi}^\mathbf{S}(\theta_1(t), \theta_2(t), \mathbf{x})) + p_\mathbf{w} \circ \mathbf{\Psi}^\mathbf{S}(\theta_1(t), \theta_2(t), \mathbf{x}). \end{cases}\end{aligned}$$

The quadruplet $(\mathbf{u}, p, \theta_1, \theta_2)$ is solution to (1.25) where

$$\mathbf{h} = \mathcal{K}_\delta\Big(\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi}^\mathbf{S}}(\theta_1(t), \theta_2(t), .))^T \mathbf{u}(t, \mathbf{\Phi}^\mathbf{S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.), \theta_1(t), \theta_2(t), \dot{\theta}_1(t), \dot{\theta}_2(t)\Big).$$

Moreover, for ε small enough and for every $t > 0$, we have the estimate

$$\begin{aligned}\|\mathbf{u}(t, \mathbf{\Phi}^\mathbf{S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}\|_{\mathbf{H}^1(\mathscr{F}_s)} &\leq \|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}^\mathbf{S}}(\theta_1(t), \theta_2(t), .)) - \mathrm{I}\|_{\mathbf{H}^2(\mathscr{F}_s)} \|\mathbf{v}(t) + \mathbf{w}\|_{\mathbf{H}^1(\mathscr{F}_s)} \\ &\qquad + \|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)} \\ &\leq K(|(\theta_1(t), \theta_2(t))| + \|\mathbf{v}(t)\|_{\mathbf{H}^1(\mathscr{F}_s)}).\end{aligned}$$

Hence the estimate (2.73) implies that

$$\forall t > 0, \quad \|\mathbf{u}(t, \mathbf{\Phi}^\mathbf{S}(\theta_1(t), \theta_2(t), .)) - \mathbf{w}(.)\|_{\mathbf{H}^1(\mathscr{F}_s)} + |\theta_1(t)| + |\theta_2(t)| + |\dot{\theta}_1(t)| + |\dot{\theta}_2(t)| \leq Ce^{-\delta t}.$$

This concludes the proof of Theorem 2.1.5. $\square$

Chapitre 3

Simulations numériques du système d'interaction fluide–structure

3.1 Introduction

In this chapter, we study the stabilization of a finite dimensional system that corresponds to the semi–discretization in space of an infinite dimensional fluid–structure interaction problem. The continuous problem has already been studied in Chapters 1 and 2, it is recalled hereafter. The strategy used to stabilize the discretized problem is the same as the one used for the continuous problem. A similar approach to ours has been investigated in [85], the main difference is that it uses a different reduced model obtained by a 'balanced truncation model reduction'. Other similar studies have already been led for the Navier–Stokes equations [2] and for a fluid–structure interaction problem [116]. Contrary to this latter work, in which the computations are done in a fixed domain, all computations are run in the actual domain of the fluid. The method used is a fictitious domain approach.

3.1.1 Modelling of the problem

We want to study the numerical approximation of the stabilization of a fluid–structure interaction problem. The structure represents a wing airfoil that is immersed in a wind tunnel Ω (see Fig. 3.1).

The fluid is modelled by the Navier–Stokes equations, it fulfils mixed boundary conditions on the boundary of the wind tunnel and an adherence boundary condition with the structure. The structure is represented by two scalar parameters θ_1 and θ_2 that can be respectively thought of as the pitch angle of the wing airfoil and the aileron deflection angle (see Fig. 3.2).

These angles stay within an admissible domain $\mathbb{D}_\Theta$ that is an open connected subset of $\mathbb{R}^2$, we also assume that $(0,0) \in \mathbb{D}_\Theta$. The structure equations are given by a virtual work principle (see Chapter 1).

The domain filled by the fluid depends on the parameters of the structure $(\theta_1(t), \theta_2(t)) \in \mathbb{D}_\Theta$, it is an open set denoted $\mathscr{F}(\theta_1(t), \theta_2(t))$. The domain occupied by the structure at time t is

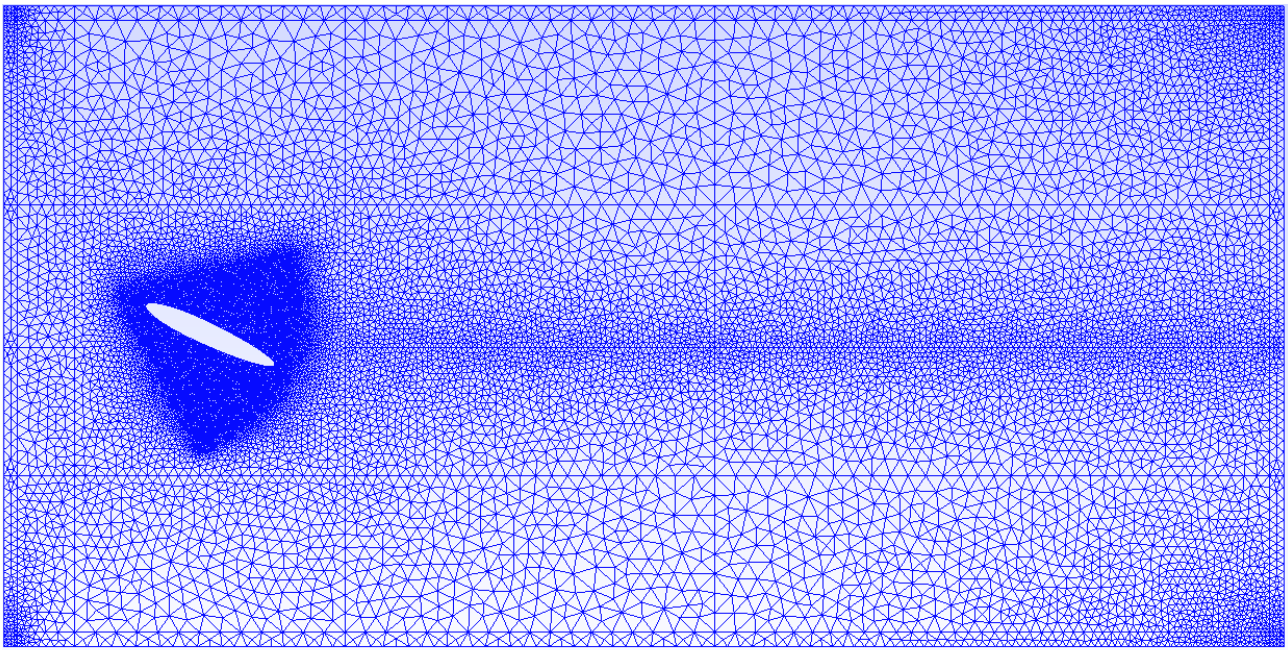


FIGURE 3.1 – The domain Ω and the discretization of $\mathscr{F}(\theta_1, \theta_2)$ for the numerical simulations.

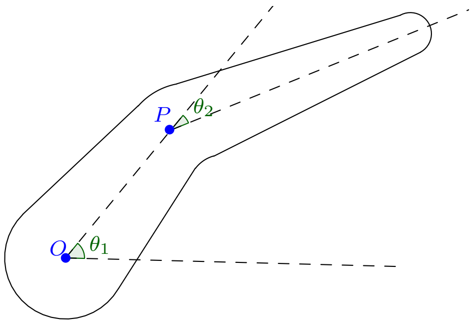


FIGURE 3.2 – The considered structure.

a closed set denoted $S(\theta_1(t), \theta_2(t))$. For every value of $\theta_1(t)$ and $\theta_2(t)$, the global domain Ω is divided between the fluid and the structure, i.e. $\Omega = S(\theta_1(t), \theta_2(t)) \cup \mathscr{F}(\theta_1(t), \theta_2(t))$.

In the present study, $\Omega = (0, L) \times (0, 1)$ is a rectangular domain. We split its boundary into three parts $\partial\Omega = \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{w}} \cup \Gamma_{\mathrm{N}}$. Inflow boundary conditions are imposed on $\Gamma_{\mathrm{i}} = \{0\} \times (0, 1)$, homogeneous Dirichlet boundary conditions are imposed on $\Gamma_{\mathrm{w}} = (0, L) \times \{0, 1\}$ and Neumann boundary conditions are imposed on $\Gamma_{\mathrm{N}} = \{L\} \times (0, 1)$. We also denote $\Gamma_{\mathrm{D}} = \overline{\Gamma_{\mathrm{i}}} \cup \Gamma_{\mathrm{w}}$ the part of $\partial\Omega$ where Dirichlet boundary conditions are imposed.

In order to take into account the deformations of the structure, we consider a function $\mathbf{X} \in \mathscr{C}^\infty(\mathbb{D}_\Theta; \mathbf{W}^{3,\infty}(S(0, 0)))$ such that for every $(\beta_1, \beta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\beta_1, \beta_2, \mathbf{y})$ is the point in $S(\beta_1, \beta_2)$ that corresponds to the particule of matter that is present in $\mathbf{y}$ in the reference configuration $S(0, 0)$.

Moreover, for every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, $\mathbf{X}(\theta_1, \theta_2, .)$ is a diffeomorphism, whose inverse is denoted $\mathbf{Y}(\theta_1, \theta_2, .)$. An instance of such a function is given in Section 3.5.1.1 and, in this chapter, we make the same assumptions on $\mathbf{X}$ as in the introduction of the manuscript, see (2)–(7).

In the sequel, we denote respectively $\dot{\theta}_j$ and $\ddot{\theta}_j$ the first and second derivatives of θ_j with respect to time. The considered system of equations is the following one

$$
\begin{cases}
\dfrac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x})+(\mathbf{u}(t,\mathbf{x})\cdot\nabla)\mathbf{u}(t,\mathbf{x})-\operatorname{div}\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))=\mathbf{f}_{\mathscr{F}}(\mathbf{x}), & t\in(0,T),\quad \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\operatorname{div}\,\mathbf{u}(t,\mathbf{x})=0, & t\in(0,T),\quad \mathbf{x}\in\mathscr{F}(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x})=\displaystyle\sum_{j=1}^{2}\dot\theta_j(t)\partial_{\theta_j}\mathbf{X}(\theta_1(t),\theta_2(t),\mathbf{Y}(\theta_1(t),\theta_2(t),\mathbf{x})), & t\in(0,T),\quad \mathbf{x}\in\partial S(\theta_1(t),\theta_2(t)),\\
\mathbf{u}(t,\mathbf{x})=\mathbf{u}^i(\mathbf{x})+\mathbf{u}^p(t,\mathbf{x}), & t\in(0,T),\quad \mathbf{x}\in\Gamma_{\mathrm{i}},\\
\mathbf{u}(t,\mathbf{x})=0, & t\in(0,T),\quad \mathbf{x}\in\Gamma_{\mathrm{w}},\\
\sigma_F(\mathbf{u}(t,\mathbf{x}),p(t,\mathbf{x}))\mathbf{n}(\mathbf{x})=0, & t\in(0,T),\quad \mathbf{x}\in\Gamma_{\mathrm{N}},\\
\mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x}), & \mathbf{x}\in\mathscr{F}(\theta_{1,0},\theta_{2,0}),\\
\mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix}=\mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)+\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}) & \\
\qquad\qquad +\mathbf{f_s}-k\begin{pmatrix}\theta_1-\xi_1\\ \theta_2-\xi_2\end{pmatrix}+\mathbf{h}, & t\in(0,T),\\
\theta_1(0)=\theta_{1,0},\qquad \theta_2(0)=\theta_{2,0}, & \\
\dot\theta_1(0)=\omega_{1,0},\qquad \dot\theta_2(0)=\omega_{2,0}. &
\end{cases}
\tag{3.1}
$$

In the previous system, $\mathbf{u}$ and p represent respectively the velocity and the pressure of the fluid that is contained in $\mathscr{F}(\theta_1(t),\theta_2(t))$. We have considered the source terms $\mathbf{f}_{\mathscr{F}}\in\mathbf{W}^{1,\infty}(\Omega)$ and $\mathbf{f_s}\in\mathbb{R}^2$, a nonhomogeneous inflow boundary datum $\mathbf{u}^i+\mathbf{u}^p$ and initial data $\mathbf{u}_0\in\mathbf{H}^1(\mathscr{F}(\theta_{1,0},\theta_{2,0}))$ for the fluid and $(\theta_{1,0},\theta_{2,0})\in\mathbb{D}_\Theta$, $(\omega_{1,0},\omega_{2,0})\in\mathbb{R}^2$ for the structure. The perturbation $\mathbf{u}^p$ is used in the sequel to destabilize a stationary state. We consider a control $\mathbf{h}\in\mathrm{L}^2(0,T;\mathbb{R}^2)$ modelling a force acting on the structure. We denote respectively $\mathbf{n}$ and $\mathbf{n}_{\theta_1,\theta_2}$ the outward unitary normals to Ω and $\mathscr{F}(\theta_1,\theta_2)$. Moreover, we have defined

$$\sigma_F(\mathbf{u},p)=\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-p\mathrm{I},$$

the Cauchy stress tensor of the fluid where $\nu>0$ is the fluid viscosity.

In the equation of the structure, we use $k\geq 0$ a rigidity coefficient that make the state of the structure tend to the state (ξ_1,ξ_2) around which we try to stabilize the structure. Moreover, we have the following terms

$$\mathcal{M}_{\theta_1,\theta_2}=\begin{pmatrix}(\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S\\ (\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S & (\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S\end{pmatrix}\in\mathbb{R}^{2\times 2},$$

$$\mathbf{M_I}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)=\begin{pmatrix}-(\dot\theta_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2)+2\dot\theta_1\dot\theta_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2)+\dot\theta_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2))_S\\ -(\dot\theta_1^2\partial_{\theta_1\theta_1}\mathbf{X}(\theta_1,\theta_2)+2\dot\theta_1\dot\theta_2\partial_{\theta_1\theta_2}\mathbf{X}(\theta_1,\theta_2)+\dot\theta_2^2\partial_{\theta_2\theta_2}\mathbf{X}(\theta_1,\theta_2),\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2))_S\end{pmatrix}\in\mathbb{R}^2,$$

where $(.,.)_S$ is the scalar product

$$(\mathbf{\Phi},\mathbf{\Psi})_S=\int_{S(0,0)}\rho\mathbf{\Phi}(\mathbf{y})\cdot\mathbf{\Psi}(\mathbf{y})\,\mathrm{d}\mathbf{y},\tag{3.2}$$

and $\rho>0$ is a constant that represents the mass per unit volume of the structure. Moreover,

$$\mathbf{M_A}(\theta_1,\theta_2,-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2})=\begin{pmatrix}\displaystyle\int_{\partial S(\theta_1,\theta_2)}-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}\cdot\partial_{\theta_1}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x\\ \displaystyle\int_{\partial S(\theta_1,\theta_2)}-\sigma_F(\mathbf{u},p)\mathbf{n}_{\theta_1,\theta_2}\cdot\partial_{\theta_2}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x\end{pmatrix}\in\mathbb{R}^2.$$

Remark 3.1.1. The scalar k can also be replaced by a positive matrix.

Existence and uniqueness of strong solutions to (3.1) have been proven for $k = 0$ in Chapter 1. The reader can report to this study to get further information about the modelling. The results proven in Chapter 1 hold for $k > 0$ because it induces a bounded perturbation of the original semigroup, hence all the proofs can be adapted.

3.1.2 Stabilization of the continuous problem

The stabilization of the solution $(\mathbf{u}, p, \theta_1, \theta_2)$ to (3.1) around a target state $(\mathbf{w}, p_{\mathbf{w}}, \xi_1, \xi_2)$ has been studied for $k = 0$ in Chapter 2. Let $(\xi_1, \xi_2) \in \mathbb{D}_\Theta$ be the parameters corresponding to the target position for the structure. We denote $\mathscr{F}_s = \mathscr{F}(\xi_1, \xi_2)$ and $S_s = S(\xi_1, \xi_2)$ the target fluid and structure domains.

We consider the nonhomogeneous boundary datum $\mathbf{u}^i$ in

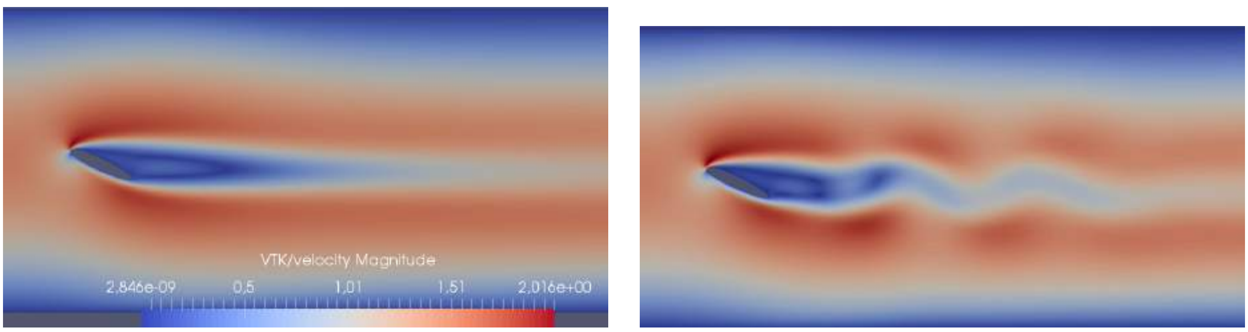
$$\mathbf{U}^i = \left\{ \begin{array}{ll} \mathbf{u}^i \in \mathbf{H}^{3/2}(\Gamma_{\rm i}) \ \text{ with } \ \mathbf{u}^i_{|\partial\Gamma_{\rm i}} = 0, & \displaystyle\int_0^{1/4} \frac{|\partial_{y_2} u_2^i(y_2)|^2}{y_2}\, \mathrm{d}y_2 < +\infty, \\ & \displaystyle\int_{3/4}^1 \frac{|\partial_{y_2} u_2^i(y_2)|^2}{1-y_2}\, \mathrm{d}y_2 < +\infty \end{array} \right\}.$$

We denote $\mathbf{n}_s$ the outward unitary normal to $\mathscr{F}_s$.

Let $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f}_{\mathbf{s}}) \in \mathbf{W}^{1,\infty}(\Omega) \times \mathbf{U}^i \times \mathbb{R}^2$ and $(\mathbf{w}, p_{\mathbf{w}}) \in \mathbf{H}^{3/2}(\mathscr{F}_s) \times \mathrm{H}^{1/2}(\mathscr{F}_s)$ fulfilling the following stationary equations

$$\left\{ \begin{array}{ll} -\mathrm{div}\ \sigma_F(\mathbf{w}, p_{\mathbf{w}}) = -(\mathbf{w} \cdot \nabla)\mathbf{w} + \mathbf{f}_{\mathscr{F}} & \text{in } \mathscr{F}_s, \\ \mathrm{div}\ \mathbf{w} = 0 & \text{in } \mathscr{F}_s, \\ \mathbf{w} = 0 & \text{on } \Gamma_{\rm w} \cup \partial S_s, \\ \mathbf{w} = \mathbf{u}^i & \text{on } \Gamma_{\rm i}, \\ \sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n} = 0 & \text{on } \Gamma_{\rm N}, \\ \mathbf{f}_{\mathbf{s}} = \left(\displaystyle\int_{\partial S_s} \sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{X}(\xi_1, \xi_2, \mathbf{Y}(\xi_1, \xi_2, \gamma_y))\, \mathrm{d}\gamma_y \right)_{j=1,2}. & \end{array} \right. \tag{3.3}$$

In our study, we choose $(\mathbf{f}_{\mathscr{F}}, \mathbf{u}^i, \mathbf{f}_{\mathbf{s}})$ and $(\mathbf{w}, p_{\mathbf{w}}, \xi_1, \xi_2)$ such that the solution $(\mathbf{w}, p_{\mathbf{w}}, \xi_1, \xi_2)$ of (3.3) is an unstable stationary solution of (3.1). We represent the magnitude of such a $\mathbf{w}$ in Fig. 3.3a. If a perturbation is introduced with no control, the state of the system starts to oscillate as in Fig. 3.3b. This phenomenon is known as the von Kármán vortex street.



(a) The stationary solution. (b) The open loop solution ($t = 5s$).

FIGURE 3.3 – The stationary and unstationary solutions ($\mathcal{R}e = 120$).

The goal of the study is to find a control $\mathbf{h}$, in a feedback form, able to stabilize the system (3.1) around the stationary solution with the arbitrary exponential decay rate $\delta > 0$. Providing a numerical control to a fluid–structure interaction flow is an active research area [144, 46].

In the sequel, the initial state of the problem is taken as the stationary state $\mathbf{u}_0 = \mathbf{w}$, $\theta_{1,0} = \xi_1$, $\theta_{2,0} = \xi_2$ and $\omega_{1,0} = \omega_{2,0} = 0$. The perturbation $\mathbf{u}^p$ is introduced on Γ_{i} and destabilizes the system.

A difficulty that appears in all fluid–structure interaction problems is that the fluid domain depends on (θ_1, θ_2). The usual way to overcome this difficulty is to bring the study back into a fixed domain. This is done with the help of a diffeomorphism and a change of variables. In the present study, the diffeomorphism used is an extension of $\mathbf{X}$ into the full domain Ω. We define a function $\mathbf{\Phi^S}$ by

$$\forall(\widetilde{\theta}_1, \widetilde{\theta}_2) \in \mathbb{D}_\Theta, \quad \forall \mathbf{y} \in \Omega, \quad \mathbf{\Phi^S}(\widetilde{\theta}_1, \widetilde{\theta}_2, \mathbf{y}) = \mathbf{y} + \mathcal{E}\Big(\mathbf{X}(\xi_1 + \widetilde{\theta}_1, \xi_2 + \widetilde{\theta}_2, \mathbf{Y}(\xi_1, \xi_2, .)) - \mathrm{Id}\Big)(\mathbf{y}),$$

where Id is the identity function of S_s and $\mathcal{E} : \mathbf{W}^{3,\infty}(S_s) \to \mathbf{W}^{3,\infty}(\Omega) \cap \mathbf{H}^1_0(\Omega)$ is a continuous extension operator that vanishes near $\partial\Omega$ (see Chapter 2, Lemma 2.1.3). For $\mathbb{D}_\Theta$ small enough, the function $\mathbf{\Phi^S}(\theta_1, \theta_2, .)$ is a diffeomorphism.

We study the difference between the state of the system written in the fixed domain $\mathscr{F}_s$ and the stationary solution defined in the domain $\mathscr{F}_s$. We denote

$$\forall t \in (0, T), \quad \forall \mathbf{y} \in \mathscr{F}_s, \quad \left\{ \begin{array}{l} \widetilde{\theta}_1(t) = \theta_1(t) - \xi_1, \\ \widetilde{\theta}_2(t) = \theta_2(t) - \xi_2, \\ \widetilde{\mathbf{u}}(t, \mathbf{y}) = \mathrm{cof}(\mathscr{J}_{\mathbf{\Phi^S}}(\widetilde{\theta}_1(t), \widetilde{\theta}_2(t), \mathbf{y}))^T \mathbf{u}(t, \mathbf{\Phi^S}(\widetilde{\theta}_1(t), \widetilde{\theta}_2(t), \mathbf{y})) - \mathbf{w}(\mathbf{y}), \\ \widetilde{p}(t, \mathbf{y}) = p(t, \mathbf{\Phi^S}(\widetilde{\theta}_1, \widetilde{\theta}_2, \mathbf{y})) - p_{\mathbf{w}}(\mathbf{y}), \end{array} \right.$$

where $\mathscr{J}_{\mathbf{\Phi^S}}(\widetilde{\theta}_1, \widetilde{\theta}_2, .) = \nabla_{\mathbf{y}} \mathbf{\Phi^S}(\widetilde{\theta}_1, \widetilde{\theta}_2, .)$ stands for the Jacobian matrix of $\mathbf{\Phi^S}(\widetilde{\theta}_1, \widetilde{\theta}_2, .)$. In Chapter 2, Section 2.3.1, we have proven that $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\theta}_1, \widetilde{\theta}_2)$ is solution to

$$\left\{ \begin{array}{ll} \dfrac{\partial \widetilde{\mathbf{u}}}{\partial t} + (\mathbf{w} \cdot \nabla)\widetilde{\mathbf{u}} + (\widetilde{\mathbf{u}} \cdot \nabla)\mathbf{w} - \mathbf{L_F}(\widetilde{\theta}_1, \widetilde{\theta}_2, \dot{\widetilde{\theta}}_1, \dot{\widetilde{\theta}}_2, \mathbf{y}) & \\ \qquad\qquad\qquad\qquad - \mathrm{div}\,\sigma_F(\widetilde{\mathbf{u}}, \widetilde{p}) = \mathbf{f}^{NL}(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\mathbf{u}}, \widetilde{p}) & \text{in } (0,T) \times \mathscr{F}_s, \\ \mathrm{div}\, \widetilde{\mathbf{u}} = 0 & \text{in } (0,T) \times \mathscr{F}_s, \\ \widetilde{\mathbf{u}} = \dot{\widetilde{\theta}}_1 \partial_{\theta_1} \mathbf{\Phi^S}(0, 0, \gamma_y) + \dot{\widetilde{\theta}}_2 \partial_{\theta_2} \mathbf{\Phi^S}(0, 0, \gamma_y) + \mathbf{g}^{NL}(\widetilde{\theta}_1, \widetilde{\theta}_2) & \text{on } (0,T) \times \partial S_s, \\ \widetilde{\mathbf{u}} = \mathbf{u}^p & \text{on } (0,T) \times \Gamma_{\mathrm{i}}, \\ \widetilde{\mathbf{u}} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{w}}, \\ \sigma_F(\widetilde{\mathbf{u}}, \widetilde{p})\mathbf{n} = 0 & \text{on } (0,T) \times \Gamma_{\mathrm{N}}, \\ \widetilde{\mathbf{u}}(0, \mathbf{y}) = \widetilde{\mathbf{u}}_0(\mathbf{y}) = \mathrm{cof}(\mathscr{J}_{\mathbf{\Phi^S}}(\widetilde{\theta}_{1,0}, \widetilde{\theta}_{2,0}, \mathbf{y}))^T \mathbf{u}_0(\mathbf{\Phi^S}(\widetilde{\theta}_{1,0}, \widetilde{\theta}_{2,0}, \mathbf{y})) - \mathbf{w}(\mathbf{y}) & \text{in } \mathscr{F}_s, \\ \mathcal{M}_{\xi_1, \xi_2} \begin{pmatrix} \ddot{\widetilde{\theta}}_1 \\ \ddot{\widetilde{\theta}}_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} -\sigma_F(\widetilde{\mathbf{u}}, \widetilde{p})\mathbf{n}_s \cdot \partial_{\theta_1} \mathbf{\Phi^S}(0, 0, \gamma_y)\, \mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_s} -\sigma_F(\widetilde{\mathbf{u}}, \widetilde{p})\mathbf{n}_s \cdot \partial_{\theta_2} \mathbf{\Phi^S}(0, 0, \gamma_y)\, \mathrm{d}\gamma_y \end{pmatrix} & \\ \qquad\qquad - k \begin{pmatrix} \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \end{pmatrix} + \mathbf{L_S}(\widetilde{\theta}_1, \widetilde{\theta}_2) + \mathbf{s}^{NL}(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\mathbf{u}}, \widetilde{p}) + \mathbf{h} & \text{on } (0,T), \\ \widetilde{\theta}_1(0) = \widetilde{\theta}_{1,0}, \qquad \widetilde{\theta}_2(0) = \widetilde{\theta}_{2,0}, & \\ \dot{\widetilde{\theta}}_1(0) = \widetilde{\omega}_{1,0}, \qquad \dot{\widetilde{\theta}}_2(0) = \widetilde{\omega}_{2,0}, & \end{array} \right. \tag{3.4}$$

where $\mathbf{f}^{NL}$, $\mathbf{g}^{NL}$ and $\mathbf{s}^{NL}$ are nonlinear terms given in Chapter 2, Section 2.3.1. Moreover, $\mathbf{L_F}$ and $\mathbf{L_S}$ are linear terms defined in Appendix B, $\widetilde{\theta}_{j,0} = \theta_{j,0} - \xi_j$ and $\widetilde{\omega}_{j,0} = \omega_{j,0}$.

In the sequel we focus on the linear system associated to (3.4) which is given by

$$
\begin{cases}
\dfrac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{w}\cdot\nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\mathbf{w} - \mathbf{L_F}(\widetilde{\theta}_1,\widetilde{\theta}_2,\dot{\widetilde{\theta}}_1,\dot{\widetilde{\theta}}_2,\mathbf{y}) - \operatorname{div}\,\sigma_F(\tilde{\mathbf{u}},\widetilde{p}) = 0 & \text{in } (0,T)\times\mathscr{F}_s,\\
\operatorname{div}\,\tilde{\mathbf{u}} = 0 & \text{in } (0,T)\times\mathscr{F}_s,\\
\tilde{\mathbf{u}} = \dot{\widetilde{\theta}}_1\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y) + \dot{\widetilde{\theta}}_2\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y) & \text{on } (0,T)\times\partial S_s,\\
\tilde{\mathbf{u}} = \mathbf{u}^p & \text{on } (0,T)\times\Gamma_{\mathrm{i}},\\
\tilde{\mathbf{u}} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{w}},\\
\sigma_F(\tilde{\mathbf{u}},\widetilde{p})\mathbf{n} = 0 & \text{on } (0,T)\times\Gamma_{\mathrm{N}},\\
\tilde{\mathbf{u}}(0,\mathbf{y}) = \tilde{\mathbf{u}}_0(\mathbf{y}) & \text{in } \mathscr{F}_s,\\
\mathcal{M}_{\xi_1,\xi_2}\begin{pmatrix}\ddot{\widetilde{\theta}}_1\\ \ddot{\widetilde{\theta}}_2\end{pmatrix} = \begin{pmatrix}\displaystyle\int_{\partial S_s} -\sigma_F(\tilde{\mathbf{u}},\widetilde{p})\mathbf{n}_s\cdot\partial_{\theta_1}\mathbf{\Phi^S}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\\ \displaystyle\int_{\partial S_s} -\sigma_F(\tilde{\mathbf{u}},\widetilde{p})\mathbf{n}_s\cdot\partial_{\theta_2}\mathbf{\Phi^S}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\end{pmatrix} & \\
\qquad\qquad -k\begin{pmatrix}\widetilde{\theta}_1\\ \widetilde{\theta}_2\end{pmatrix} + \mathbf{L_S}(\widetilde{\theta}_1,\widetilde{\theta}_2) + \mathbf{h} & \text{on } (0,T),\\
\widetilde{\theta}_1(0) = \tilde{\theta}_{1,0}, \qquad \widetilde{\theta}_2(0) = \tilde{\theta}_{2,0}, & \\
\dot{\widetilde{\theta}}_1(0) = \tilde{\omega}_{1,0}, \qquad \dot{\widetilde{\theta}}_2(0) = \tilde{\omega}_{2,0}. &
\end{cases}
\tag{3.5}
$$

In Chapter 2, we used a feedback operator $\mathcal{K}_\delta$ such that the control $\mathbf{h} = \mathcal{K}_\delta(\tilde{\mathbf{u}},\widetilde{\theta}_1,\widetilde{\theta}_2,\dot{\widetilde{\theta}}_1,\dot{\widetilde{\theta}}_2)$ stabilizes the problem (3.5). We used the space

$$
\mathbb{H} = \left\{\begin{array}{l} (\tilde{\mathbf{u}},\widetilde{\theta}_1,\widetilde{\theta}_2,\tilde{\omega}_1,\tilde{\omega}_2) \in \mathbf{L}^2(\mathscr{F}_s)\times\mathbb{R}^4, \quad \operatorname{div}\,\tilde{\mathbf{u}} = 0 \text{ in } \mathscr{F}_s, \quad \tilde{\mathbf{u}}\cdot\mathbf{n} = 0 \text{ on } \Gamma_{\mathrm{D}},\\ \hfill \tilde{\mathbf{u}}\cdot\mathbf{n}_s = \sum\limits_j \tilde{\omega}_j\partial_{\theta_j}\mathbf{\Phi^S}(0,0,.)\cdot\mathbf{n}_s \text{ on } \partial S_s \end{array}\right\}, \tag{3.6}
$$

and $\Pi_{\mathbb{H}}$ the orthogonal projector of $\mathbf{L}^2(\mathscr{F}_s)\times\mathbb{R}^4$ onto $\mathbb{H}$ with respect to the scalar product $(.,.)_0$ of $\mathbf{L}^2(\mathscr{F}_s)\times\mathbb{R}^4$ defined by

$$
\begin{aligned}
&\forall(\tilde{\mathbf{u}}^j,\widetilde{\theta}_1^j,\widetilde{\theta}_2^j,\tilde{\omega}_1^j,\tilde{\omega}_2^j) \in \mathbf{L}^2(\mathscr{F}_s)\times\mathbb{R}^4,\\
&\Big((\tilde{\mathbf{u}}^a,\widetilde{\theta}_1^a,\widetilde{\theta}_2^a,\tilde{\omega}_1^a,\tilde{\omega}_2^a),(\tilde{\mathbf{u}}^b,\widetilde{\theta}_1^b,\widetilde{\theta}_2^b,\tilde{\omega}_1^b,\tilde{\omega}_2^b)\Big)_0 = \int_{\mathscr{F}_s}\tilde{\mathbf{u}}^a\cdot\tilde{\mathbf{u}}^b\mathrm{d}\mathbf{y} + \begin{pmatrix}\widetilde{\theta}_1^a & \widetilde{\theta}_2^a\end{pmatrix}\begin{pmatrix}\widetilde{\theta}_1^b\\ \widetilde{\theta}_2^b\end{pmatrix}\\
&\hspace{22em} + \begin{pmatrix}\tilde{\omega}_1^a & \tilde{\omega}_2^a\end{pmatrix}\mathcal{M}_{\xi_1,\xi_2}\begin{pmatrix}\tilde{\omega}_1^b\\ \tilde{\omega}_2^b\end{pmatrix}.
\end{aligned}
\tag{3.7}
$$

We then studied the semigroup formulation of the linearized problem (3.5). We used a spectral analysis of the operator of the semigroup formulation and the resolution of a Riccati equation. Under a unique continuation property that we assumed (see Hypothesis $\boldsymbol{(\mathcal{H})}_\delta$), we have constructed a feedback operator $\mathcal{K}_\delta$ that stabilizes the continuous problem (3.5).

The detailed analysis of this stabilization problem has been done in Chapter 2. Here we would like to develop a similar strategy for the semi–discrete system obtained by approximating by a Finite Element Method (FEM in the sequel) the system (3.1).

We sum up our approach in the following steps.

- In Section 3.2, we present the matrix formulation of the semi–discrete Finite Element approximation.
- In Section 3.3, we reformulate the finite dimensional problem as a control system by eliminating the multiplier from the equations using a projector corresponding to the

discretization of $\Pi_{\mathbb{H}}$, the projector onto $\mathbb{H}$ defined in (3.6). Then, we lead a spectral analysis of the matrices obtained by the projection, we study the projection of the system onto a low dimensional space that contains all the unstable eigenvectors of the problem. We construct a stabilizing feedback matrix $\mathbf{K}_{\delta,\omega}$ by solving an Algebraic Riccati Equation.

— In Section 3.4, we study the relationships between the eigenvalue problems involving Lagrange multipliers and those without Lagrange multipliers. Using these relationships, we are able to construct the feedback law without having to use the projector which is difficult to construct numerically. We then show a more practical way to build the feedback matrix $\mathbf{K}_{\delta,\omega}$ using the solution of a low dimensional Algebraic Riccati Equation.

— In Section 3.5, we give some details on the construction of the diffeomorphisms used and we show the sprectra associated to our problem.

— In Section 3.6, we present the numerical tests that we ran using a fictitious domain method to simulate the fluid–structure interaction system (3.1) with and without control.

Note that we use a conformal mesh in Sections 3.2 to 3.5 in order to construct a feedback matrix and a non conformal one in Section 3.6 in order to run numerical simulations.

3.2 The semi–discretized approximations

In the sequel, the Dirichlet boundary conditions are implemented with Lagrange multipliers $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\rm s}, \boldsymbol{\lambda}_{\rm i}, \boldsymbol{\lambda}_{\rm w})^T$. First, we exhibit the variational formulation of (3.5) with Lagrange multipliers. Then, we discretize this variational problem in a basis of finite elements, which gives the system (3.12). For the sake of readibility we drop the tildes in $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\theta}_1, \widetilde{\theta}_2)$ that we write $(\mathbf{u}, p, \theta_1, \theta_2)$ from now on. In this section we consider the system without perturbation, i.e. $\mathbf{u}^p = 0$. The goal of the comming sections is the construction of a feedback matriw $\mathbf{K}_{\delta,\omega}$ that stabilizes the system.

3.2.1 A variational formulation of the continuous problem

We denote $\mathbf{H}^{-1/2}(\Gamma_{\rm w}) = \mathbf{H}^{-1/2}((0,L)\times\{0\})\times\mathbf{H}^{-1/2}((0,L)\times\{1\})$. The system (3.5) is equivalent to the following variational formulation :

Find $(\theta_1,\theta_2,\omega_1,\omega_2)\in \mathrm{H}^2(0,T;\mathbb{D}_\Theta)\times \mathrm{H}^1(0,T;\mathbb{R}^2)$
and $(\mathbf{u},p,\boldsymbol{\lambda}_{\rm s},\boldsymbol{\lambda}_{\rm i},\boldsymbol{\lambda}_{\rm w})\in (\mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}_s))\cap \mathrm{L}^2(0,T;\mathbf{H}^1(\mathscr{F}_s)))\times \mathrm{L}^2(0,T;\mathrm{L}^2(\mathscr{F}_s))$
$\times \mathrm{L}^2(0,T;\mathbf{H}^{-1/2}(\partial S_s)\times\mathbf{H}^{-1/2}(\Gamma_{\rm i})\times\mathbf{H}^{-1/2}(\Gamma_{\rm w}))$ such that

$$\begin{cases}\displaystyle\int_{\mathscr{F}_s}\frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{v}\,\mathrm{d}\mathbf{y}+\int_{\mathscr{F}_s}2\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T):(\nabla\mathbf{v}+\nabla\mathbf{v}^T)-p\,\mathrm{div}\,\mathbf{v}+(\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}+(\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\,\mathrm{d}\mathbf{y}\\ \displaystyle\qquad -\int_{\mathscr{F}_s}\mathbf{L}_{\mathbf{F}}(\theta_1,\theta_2,\omega_1,\omega_2,\mathbf{y})\cdot\mathbf{v}\,\mathrm{d}\mathbf{y}+\int_{\partial S_s}\boldsymbol{\lambda}_{\rm s}\cdot\mathbf{v}+\int_{\Gamma_{\rm i}}\boldsymbol{\lambda}_{\rm i}\cdot\mathbf{v}+\int_{\Gamma_{\rm w}}\boldsymbol{\lambda}_{\rm w}\cdot\mathbf{v}=0,\\ \displaystyle\int_{\mathscr{F}_s} q\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\mathbf{y}=0,\\ \displaystyle\int_{\partial S_s}\boldsymbol{\mu}_{\rm s}\cdot\left(\mathbf{u}-\sum_j\omega_j\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{x})\right)\,\mathrm{d}\gamma_y=0,\\ \displaystyle\int_{\Gamma_{\rm i}}\boldsymbol{\mu}_{\rm i}\cdot\mathbf{u}\,\mathrm{d}\gamma_y=0,\\ \displaystyle\int_{\Gamma_{\rm w}}\boldsymbol{\mu}_{\rm w}\cdot\mathbf{u}\,\mathrm{d}\gamma_y=0,\\ \text{for every } (\mathbf{v},q,\boldsymbol{\mu}_{\rm s},\boldsymbol{\mu}_{\rm i},\boldsymbol{\mu}_{\rm w}) \text{ in } \mathbf{H}^1(\mathscr{F}_s)\times\mathrm{L}^2(\mathscr{F}_s)\times\mathbf{H}^{-1/2}(\partial S_s)\times\mathbf{H}^{-1/2}(\Gamma_{\rm i})\times\mathbf{H}^{-1/2}(\Gamma_{\rm w}),\\ \mathcal{M}_{\xi_1,\xi_2}\begin{pmatrix}\dot\omega_1\\ \dot\omega_2\end{pmatrix}=\begin{pmatrix}\displaystyle\int_{\partial S_s}\boldsymbol{\lambda}_{\rm s}(\gamma_y)\cdot\partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\\ \displaystyle\int_{\partial S_s}\boldsymbol{\lambda}_{\rm s}(\gamma_y)\cdot\partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\,\mathrm{d}\gamma_y\end{pmatrix}-k\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}+\theta_1\mathbf{L}_5+\theta_2\mathbf{L}_6+\mathbf{h},\\ \dot\theta_1=\omega_1,\\ \dot\theta_2=\omega_2,\\ \mathbf{u}(0)=\mathbf{u}_0,\quad \theta_1(0)=\theta_{1,0},\quad \theta_2(0)=\theta_{2,0},\quad \omega_1(0)=\omega_{1,0},\quad \omega_2(0)=\omega_{2,0}.\end{cases}\tag{3.8}$$

Moreover, we can write $\mathbf{L}_5$ and $\mathbf{L}_6$ under the form

$$\begin{aligned}\mathbf{L}_5=\int_{\partial S_s}\boldsymbol{\lambda}_{\rm statio}\cdot\partial_{\theta_1\theta_i}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)-\sum_{k,l}\sigma_F(\mathbf{w},p_{\mathbf{w}})_{lk}(\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{k1}\partial_{\theta_i}\Phi_l^S(0,0,\gamma_y)\\ -(\nabla_{\mathbf{y}}\partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\mathbf{t}_s\cdot\mathbf{t}_s)\boldsymbol{\lambda}_{\rm statio}\cdot\partial_{\theta_i}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\\ -\nu\sum_{k,l}((\mathbf{L}_{\mathcal{G}})_{kl1}+(\mathbf{L}_{\mathcal{G}})_{lk1})(\mathbf{n}_s)_k\partial_{\theta_i}\Phi_l^S(0,0,\gamma_y)\,\mathrm{d}\gamma_y,\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}\mathbf{L}_6=\int_{\partial S_s}\boldsymbol{\lambda}_{\rm statio}\cdot\partial_{\theta_2\theta_i}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)-\sum_{k,l}\sigma_F(\mathbf{w},p_{\mathbf{w}})_{lk}(\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{k2}\partial_{\theta_i}\Phi_l^S(0,0,\gamma_y)\\ -(\nabla_{\mathbf{y}}\partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\mathbf{t}_s\cdot\mathbf{t}_s)\boldsymbol{\lambda}_{\rm statio}\cdot\partial_{\theta_i}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\\ -\nu\sum_{k,l}((\mathbf{L}_{\mathcal{G}})_{kl2}+(\mathbf{L}_{\mathcal{G}})_{lk2})(\mathbf{n}_s)_k\partial_{\theta_i}\Phi_l^S(0,0,\gamma_y)\,\mathrm{d}\gamma_y,\end{aligned}\tag{3.10}$$

where $\boldsymbol{\lambda}_{\rm statio}=-\sigma_F(\mathbf{w},p_{\mathbf{w}})\mathbf{n}_s$ and $\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}}$, $\mathbf{L}_{\mathcal{G}}$ are defined in Appendix B.

Remark 3.2.1. We have the relation $\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)=\partial_{\theta_j}\mathbf{X}(\xi_1,\xi_2,\mathbf{Y}(\xi_1,\xi_2,.))$ on ∂S_s, this eases the computation of $\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)$ on ∂S_s.

3.2.2 Discretization of the variational formulation

In this section we derive the semi–discretized problem associated to (3.8).

3.2.2.1 The space discretization

We consider the finite element spaces :

$$V_h \subset \mathbf{H}^1(\mathscr{F}_s), \qquad Q_h \subset \mathrm{L}^2(\mathscr{F}_s), \qquad W_h \subset \mathbf{H}^{-1/2}(\partial S_s) \times \mathbf{H}^{-1/2}(\Gamma_{\mathrm{i}}) \times \mathbf{H}^{-1/2}(\Gamma_{\mathrm{w}}).$$

We denote $N_{\mathbf{u}} = \dim(V_h)$, $N_p = \dim(Q_h)$ and $N_{\boldsymbol{\lambda}} = \dim(W_h)$ and $\mathbf{u}_h \in V_h$, $p_h \in Q_h$, $\boldsymbol{\lambda}_h = (\boldsymbol{\lambda}_{\mathrm{s},h}, \boldsymbol{\lambda}_{\mathrm{i},h}, \boldsymbol{\lambda}_{\mathrm{w},h}) \in W_h$ the semi–discretization in space of $\mathbf{u}$, p and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\mathrm{s}}, \boldsymbol{\lambda}_{\mathrm{i}}, \boldsymbol{\lambda}_{\mathrm{w}})$ respectively. It is solution to the following variational problem.

Find $(\theta_1, \theta_2, \omega_1, \omega_2) \in \mathrm{H}^2(0,T;\mathbb{D}_\Theta) \times \mathrm{H}^1(0,T;\mathbb{R}^2)$
and $(\mathbf{u}_h, p_h, \boldsymbol{\lambda}_{\mathrm{s},h}, \boldsymbol{\lambda}_{\mathrm{i},h}, \boldsymbol{\lambda}_{\mathrm{w},h}) \in \mathrm{H}^1(0,T;V_h) \times \mathrm{L}^2(0,T;Q_h) \times \mathrm{L}^2(0,T;W_h)$ such that

$$\begin{cases}
\displaystyle\int_{\mathscr{F}_s} \frac{\mathrm{d}\mathbf{u}_h}{\mathrm{d}t} \cdot \mathbf{v}_h + 2\nu(\nabla \mathbf{u}_h + \nabla \mathbf{u}_h^T) : (\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T) - p_h \operatorname{div} \mathbf{v}_h + (\mathbf{w}\cdot\nabla)\mathbf{u}_h \cdot \mathbf{v}_h + (\mathbf{u}_h \cdot \nabla)\mathbf{w}\cdot \mathbf{v}_h \, \mathrm{d}\mathbf{y} \\
\qquad \displaystyle - \int_{\mathscr{F}_s} \mathbf{L}_{\mathbf{F}}(\theta_1, \theta_2, \omega_1, \omega_2, \mathbf{y}) \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{y} + \int_{\partial S_s} \boldsymbol{\lambda}_{\mathrm{s},h} \cdot \mathbf{v}_h + \int_{\Gamma_{\mathrm{i}}} \boldsymbol{\lambda}_{\mathrm{i},h} \cdot \mathbf{v}_h + \int_{\Gamma_{\mathrm{w}}} \boldsymbol{\lambda}_{\mathrm{w},h} \cdot \mathbf{v}_h = 0, \\
\displaystyle\int_{\mathscr{F}_s} q_h \operatorname{div} \mathbf{u}_h \, \mathrm{d}\mathbf{y} = 0, \\
\displaystyle\int_{\partial S_s} \boldsymbol{\mu}_{\mathrm{s},h} \cdot \left(\mathbf{u}_h - \sum_j \omega_j \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{x}) \right) \mathrm{d}\gamma_y = 0, \\
\displaystyle\int_{\Gamma_{\mathrm{i}}} \boldsymbol{\mu}_{\mathrm{i},h} \cdot \mathbf{u}_h \, \mathrm{d}\gamma_y = 0, \\
\displaystyle\int_{\Gamma_{\mathrm{w}}} \boldsymbol{\mu}_{\mathrm{w},h} \cdot \mathbf{u}_h \, \mathrm{d}\gamma_y = 0, \\
\text{for every } (\mathbf{v}_h, q_h, \boldsymbol{\mu}_{\mathrm{s},h}, \boldsymbol{\mu}_{\mathrm{i},h}, \boldsymbol{\mu}_{\mathrm{w},h}) \text{ in } V_h \times Q_h \times W_h, \\
\mathcal{M}_{\xi_1,\xi_2} \begin{pmatrix} \dot\omega_1 \\ \dot\omega_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\partial S_s} \boldsymbol{\lambda}_{\mathrm{s},h}(\gamma_y) \cdot \partial_{\theta_1} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \\ \displaystyle\int_{\partial S_s} \boldsymbol{\lambda}_{\mathrm{s},h}(\gamma_y) \cdot \partial_{\theta_2} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \, \mathrm{d}\gamma_y \end{pmatrix} - k \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + \theta_1 \mathbf{L}_5 + \theta_2 \mathbf{L}_6 + \mathbf{h}, \\
\dot\theta_1 = \omega_1, \\
\dot\theta_2 = \omega_2, \\
\mathbf{u}_h(0) = \mathbf{u}_{0,h}, \;\; \theta_1(0) = \theta_{1,0}, \;\; \theta_2(0) = \theta_{2,0}, \;\; \omega_1(0) = \omega_{1,0}, \;\; \omega_2(0) = \omega_{2,0}.
\end{cases} \tag{3.11}$$

Remark 3.2.2. The variational formulation (3.8) is strong in time and weak in space. Since we use finite elements in space and finite differences in time, this variational formulation is enough. The same remark applies to the other variational formulations in the sequel. For variational formulations of the Navier–Stokes equations that are weak in space and in time, the reader can report for instance to [35, p.348].

3.2.2.2 The matrix system

We introduce $(\mathcal{U}_k)_k$, $(\mathcal{P}_k)_k$ and $(\mathcal{W}_k)_k$ respectively the basis functions of the spaces V_h, Q_h and W_h, so we have the decompositions

$$\mathbf{u}_h = \sum_{k=1}^{N_{\mathbf{u}}} U_k \mathcal{U}_k, \qquad p_h = \sum_{k=1}^{N_p} P_k \mathcal{P}_k, \qquad \boldsymbol{\lambda}_h = \sum_{k=1}^{N_{\boldsymbol{\lambda}}} \Lambda_k \mathcal{W}_k.$$

We denote $\mathbf{U} = (U_k)_{k=1..N_{\mathbf{u}}}$, $\mathbf{P} = (P_k)_{k=1..N_p}$ and $\mathbf{\Lambda} = (\Lambda_k)_{k=1..N_{\boldsymbol{\lambda}}}$ the coordinates of $\mathbf{u}_h$, p_h and $\boldsymbol{\lambda}_h$.

We denote $\mathbf{w}_h$ the numerical approximation of $\mathbf{w}$ and h_1, h_2 the coordinates of the control $\mathbf{h}$. Then, the system (3.11) can be rewriten as follows

$$\begin{pmatrix} M_{\mathbf{uu}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathcal{M}_{\xi_1,\xi_2}$}} \\ 0 & 0 & 0 & 0 & 0 & & \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Lambda} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix}$$
$$= \begin{pmatrix} A_{\mathbf{uu}} & A_{\mathbf{u}p} & A_{\mathbf{u}\boldsymbol{\lambda}} & A_{\mathbf{u}\theta_1} & A_{\mathbf{u}\theta_2} & A_{\mathbf{u}\omega_1} & A_{\mathbf{u}\omega_2} \\ A_{\mathbf{u}p}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{\mathbf{u}\boldsymbol{\lambda}}^T & 0 & 0 & 0 & 0 & A_{\boldsymbol{\lambda}\omega_1} & A_{\boldsymbol{\lambda}\omega_2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & A_{\boldsymbol{\lambda}\omega_1}^T & \multirow{2}{*}{$\mathbf{L}_5 - \begin{pmatrix} k \\ 0 \end{pmatrix}$} & \multirow{2}{*}{$\mathbf{L}_6 - \begin{pmatrix} 0 \\ k \end{pmatrix}$} & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\lambda}\omega_2}^T & & & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \\ \boldsymbol{\Lambda} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ h_1 \\ h_2 \end{pmatrix}, \tag{3.12}$$

and $\mathbf{u}_h(0) = \mathbf{u}_{0,h}$, $\theta_1(0) = \theta_{1,0}$, $\theta_2(0) = \theta_{2,0}$, $\omega_1(0) = \omega_{1,0}$, $\omega_2(0) = \omega_{2,0}$, where the matrices involved are given by :

$$(M_{\mathbf{uu}})_{jk} = \int_{\mathscr{F}_s} \mathcal{U}_j \cdot \mathcal{U}_k \, \mathrm{d}\mathbf{y},$$

$$(A_{\mathbf{uu}})_{jk} = \int_{\mathscr{F}_s} -2\nu(\nabla\mathcal{U}_j + \nabla\mathcal{U}_j^T) : (\nabla\mathcal{U}_k + \nabla\mathcal{U}_k^T) \, \mathrm{d}\mathbf{y} - \int_{\mathscr{F}_s} (\mathbf{w}_h \cdot \nabla)\mathcal{U}_k \cdot \mathcal{U}_j + (\mathcal{U}_k \cdot \nabla)\mathbf{w}_h \cdot \mathcal{U}_j \, \mathrm{d}\mathbf{y},$$

$$(A_{\mathbf{u}p})_{jk} = \int_{\mathscr{F}_s} \mathcal{P}_k \mathrm{div}\,(\mathcal{U}_j) \, \mathrm{d}\mathbf{y}, \qquad (A_{\mathbf{u}\boldsymbol{\lambda}})_{jk} = \int_{\partial S_s \cup \Gamma_{\mathrm{D}}} -\mathcal{U}_j \cdot \mathcal{W}_k \, \mathrm{d}\gamma_y,$$

$$(A_{\mathbf{u}\theta_1})_j = \int_{\mathscr{F}_s} \mathbf{L}_1(\mathbf{y}) \cdot \mathcal{U}_j(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \qquad (A_{\mathbf{u}\theta_2})_j = \int_{\mathscr{F}_s} \mathbf{L}_2(\mathbf{y}) \cdot \mathcal{U}_j(\mathbf{y}) \, \mathrm{d}\mathbf{y},$$

$$(A_{\mathbf{u}\omega_1})_j = \int_{\mathscr{F}_s} \mathbf{L}_3(\mathbf{y}) \cdot \mathcal{U}_j(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \qquad (A_{\mathbf{u}\omega_2})_j = \int_{\mathscr{F}_s} \mathbf{L}_4(\mathbf{y}) \cdot \mathcal{U}_j(\mathbf{y}) \, \mathrm{d}\mathbf{y},$$

$$(A_{\boldsymbol{\lambda}\omega_1})_j = \int_{\partial S_s} \partial_{\theta_1}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \cdot \mathcal{W}_j(\gamma_y) \, \mathrm{d}\gamma_y, \qquad (A_{\boldsymbol{\lambda}\omega_2})_j = \int_{\partial S_s} \partial_{\theta_2}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \cdot \mathcal{W}_j(\gamma_y) \, \mathrm{d}\gamma_y,$$

where $\mathbf{L}_1$–$\mathbf{L}_4$ are defined in Appendix B.

The system (3.12) can also be rewriten in a more compact form as

$$\begin{cases} M_{\mathbf{zz}} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} = A_{\mathbf{zz}}\mathbf{z} + A_{\mathbf{z}\boldsymbol{\eta}}\boldsymbol{\eta} + B\mathbf{h}, \\ A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z} = 0, \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{3.13}$$

where

$$\mathbf{z} = \begin{pmatrix} \mathbf{U} \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{pmatrix}, \quad \boldsymbol{\eta} = \begin{pmatrix} \mathbf{P} \\ \boldsymbol{\Lambda} \end{pmatrix}, \quad \mathbf{z}_0 = \begin{pmatrix} \mathbf{U}_0 \\ \theta_{1,0} \\ \theta_{2,0} \\ \omega_{1,0} \\ \omega_{2,0} \end{pmatrix},$$

with $\mathbf{U}_0$ the coordinates of $\mathbf{u}_{0,h}$ and

$$M_{\mathbf{zz}} = \begin{pmatrix} M_{\mathbf{uu}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\mathcal{M}_{\xi_1,\xi_2}$}} \\ 0 & 0 & 0 & & \end{pmatrix}, \qquad A_{\mathbf{zz}} = \begin{pmatrix} A_{\mathbf{uu}} & A_{\mathbf{u}\theta_1} & A_{\mathbf{u}\theta_2} & A_{\mathbf{u}\omega_1} & A_{\mathbf{u}\omega_2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \mathbf{L}_5 - \begin{pmatrix} k \\ 0 \end{pmatrix} & \mathbf{L}_6 - \begin{pmatrix} 0 \\ k \end{pmatrix} & 0 & 0 \\ 0 & & & 0 & 0 \end{pmatrix},$$

$$A_{\mathbf{z}\boldsymbol{\eta}} = \begin{pmatrix} A_{\mathbf{u}p} & A_{\mathbf{u}\boldsymbol{\lambda}} \\ 0 & 0 \\ 0 & 0 \\ 0 & A^T_{\boldsymbol{\lambda}\omega_1} \\ 0 & A^T_{\boldsymbol{\lambda}\omega_2} \end{pmatrix}, \qquad A_{\boldsymbol{\eta}\mathbf{z}} = A^T_{\mathbf{z}\boldsymbol{\eta}} = \begin{pmatrix} A^T_{\mathbf{u}p} & 0 & 0 & 0 & 0 \\ A^T_{\mathbf{u}\boldsymbol{\lambda}} & 0 & 0 & A_{\boldsymbol{\lambda}\omega_1} & A_{\boldsymbol{\lambda}\omega_2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote $N_z = N_{\mathbf{u}} + 4$ the number of state variables of this problem, where 4 accounts for the degrees of freedom of the structure. We also denote $N_\eta = N_p + N_{\boldsymbol{\lambda}}$ the number of multiplier degrees of freedom. Finally, $N_{\text{tot}} = N_z + N_\eta$ is the total number of degrees of freedom of the system.

We assume that $N_{\mathbf{u}} > N_p + N_{\boldsymbol{\lambda}}$ and the following inf–sup condition

$$\inf_{\substack{(p_h,\boldsymbol{\lambda}_h)\in Q_h\times W_h \\ (p_h,\boldsymbol{\lambda}_h)\neq(0,0)}} \sup_{\substack{\mathbf{u}_h\in V_h \\ \mathbf{u}_h\neq 0}} \frac{\displaystyle\int_{\mathscr{F}_s} p_h \text{div}\, \mathbf{u}_h \,\mathrm{d}\mathbf{y} + \int_{\Gamma_{\mathrm{D}}\cup\partial S_s} \boldsymbol{\lambda}_h \cdot \mathbf{u}_h \,\mathrm{d}\gamma_y}{\|\mathbf{u}_h\|_{V_h} \quad \|(p_h,\boldsymbol{\lambda}_h)\|_{Q_h\times W_h}} \geq c, \tag{3.14}$$

where $c > 0$ is a mesh independent constant. The approximation spaces chosen in Section 3.5.2 satisfy this property.

Remark 3.2.3. The inf–sup condition with a mesh independent constant $c > 0$ is a classical condition to prevent some numerical instabilities or locking effets like the checkerboard instability [60, p.186]. This is a classical condition to solve the Navier–Stokes equations [81, 36].

3.3 Feedback stabilization of the linearized system

In this section, we build a stabilization control law for (3.13) under a feedback form. The strategy followed is the one of the continuous case. We get rid of the multipliers in Sections 3.3.1–3.3.2, so that we obtain an ordinary differential equation for the direct and the adjoint systems. We decompose the direct and the adjoint operators in a basis of eigenvectors and generalized eigenvectors. Then, the problem is projected on its unstable space. Finally, we find a feedback control law that stabilizes (3.13) by solving an Algebraic Riccati Equation for the projected system on the unstable space.

Remark 3.3.1. In the oncoming analysis we do not use the fact that $A_{\mathbf{z}\boldsymbol{\eta}} = A^T_{\boldsymbol{\eta}\mathbf{z}}$. Using this property would simplify the study, however we prefer to treat the more general case in which it is not fulfilled. Note that the case $A_{\mathbf{z}\boldsymbol{\eta}} \neq A^T_{\boldsymbol{\eta}\mathbf{z}}$ occurs in some studies, see for instace [116].

3.3.1 The projected direct system

The goal of this section is to rewrite the system (3.13) as a system satisfied by $\mathbf{z}$ in which the multiplier $\boldsymbol{\eta}$ is eliminated and to characterize the multiplier $\boldsymbol{\eta}$ in terms of the state and the data.

We define

$$\mathbb{H}_h = \mathrm{Ker}(A_{\boldsymbol{\eta}\mathbf{z}}). \tag{3.15}$$

Lemma 3.3.2. *The projector* Π *of* $\mathbb{R}^{N_z}$ *onto* $\mathbb{H}_h$ *parallel to* $\mathrm{Im}(M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})$ *is defined by*

$$\Pi = \mathrm{I} - M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}.$$

In the same way, the projector Π^T *of* $\mathbb{R}^{N_z}$ *onto* $\mathrm{Ker}(A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1})$ *parallel to* $\mathrm{Im}(A_{\boldsymbol{\eta}\mathbf{z}}^T)$ *is defined by*

$$\Pi^T = \mathrm{I} - A_{\boldsymbol{\eta}\mathbf{z}}^T(A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1}A_{\boldsymbol{\eta}\mathbf{z}}^T)^{-1}A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1}.$$

Remark 3.3.3. In terms of the matrices defined above, the condition (3.14) implies that $(A_{\mathbf{u}p} \;\; A_{\mathbf{u}\boldsymbol{\lambda}})$ is full ranked. Hence, $A_{\mathbf{z}\boldsymbol{\eta}}$ and $A_{\boldsymbol{\eta}\mathbf{z}}$ are full ranked. The matrix $M_{\mathbf{zz}}$ is a mass matrix and is then invertible. This implies that $A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}$ is invertible and the projector Π is well defined.

Proof of Lemma 3.3.2. • We show, with a direct computation, that $(\Pi)^2 = \Pi$, so that Π is a projector of $\mathbb{R}^{N_z}$.
• Let $\mathbf{z} \in \mathbb{H}_h = \mathrm{Ker}(A_{\boldsymbol{\eta}\mathbf{z}})$, we then have $\Pi\mathbf{z} = \mathbf{z}$, hence $\mathbb{H}_h \subset \mathrm{Im}(\Pi)$.
• Let $\mathbf{z} \in \mathrm{Im}(\Pi)$. Since Π is a projector, we have $\mathbf{z} = \Pi\mathbf{z}$, then $M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z} = 0$. We compose this equality by $A_{\boldsymbol{\eta}\mathbf{z}}$, we get $A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z} = 0$ and $\mathrm{Im}(\Pi) = \mathrm{Ker}(A_{\boldsymbol{\eta}\mathbf{z}}) = \mathbb{H}_h$.
• Let $\mathbf{z} \in \mathrm{Im}(M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})$, there exists $\boldsymbol{\eta} \in \mathbb{R}^{N_\eta}$ such that $\mathbf{z} = M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}\boldsymbol{\eta}$. We get $\Pi\mathbf{z} = \Pi M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}\boldsymbol{\eta} = 0$. Then, we have $\mathrm{Im}(M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}) \subset \mathrm{Ker}(\Pi)$.
• Let $\mathbf{z}\in\mathrm{Ker}(\Pi)$, then we have $\mathbf{z}=M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}$. We get $\mathrm{Ker}(\Pi)=\mathrm{Im}(M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})$.
• The proof on Π is complete, we prove the properties on Π^T in a similar way. □

The projection Π is an approximation of the projector $\Pi_{\mathbb{H}}$ and $\mathbb{H}_h$ is an approximation of the space $\mathbb{H}$ defined in (3.6).

Remark 3.3.4. We a priori have $\Pi \neq \Pi^T$.

Lemma 3.3.5. *A pair* $(\mathbf{z}, \boldsymbol{\eta})$ *is solution to* (3.13) *if and only if it is solution to the following system*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Pi\mathbf{z} = \mathbb{A}\Pi\mathbf{z} + \mathbb{B}\mathbf{h}, \\ (\mathrm{I}-\Pi)\mathbf{z} = 0, \\ \boldsymbol{\eta} = -(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{zz}}\mathbf{z} - (A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}B\mathbf{h}, \end{cases} \tag{3.16}$$

where

$$\mathbb{A} = \Pi M_{\mathbf{zz}}^{-1}A_{\mathbf{zz}}, \qquad \textit{and} \qquad \mathbb{B} = \Pi M_{\mathbf{zz}}^{-1}B. \tag{3.17}$$

Proof. • Let $(\mathbf{z}, \boldsymbol{\eta})\in\mathbb{R}^{N_z}\times\mathbb{R}^{N_\eta}$ fulfil the system (3.13). We have $\mathrm{I}-\Pi = M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}$, then equation $(3.13)_2$ implies $(3.16)_2$. Now, we multiply $(3.13)_1$ by $\Pi M_{\mathbf{zz}}^{-1}$ and use $\mathbf{z} = \Pi\mathbf{z}$ and $\Pi M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}} = 0$, we get $(3.16)_1$. Finally, we multiply $(3.13)_1$ by $(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}$, we use $A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}' = 0$ and we get $(3.16)_3$.

• Let $\mathbf{z} \in \mathbb{R}^{N_z}$ fulfil $(3.16)_1$–$(3.16)_2$ and let $\boldsymbol{\eta} \in \mathbb{R}^{N_\eta}$ be given by $(3.16)_3$. The equation $(3.16)_2$ implies $(3.13)_2$. The equation $(3.16)_3$ implies that $M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}\boldsymbol{\eta}+(\mathrm{I}-\Pi)M_{\mathbf{zz}}^{-1}A_{\mathbf{zz}}\mathbf{z}+(\mathrm{I}-\Pi)M_{\mathbf{zz}}^{-1}B\mathbf{h} = 0$, which combined with $(3.16)_1$ yields $(3.13)_1$. □

The following lemma is used to prove some properties in Section 3.4

Lemma 3.3.6. *If* $\mathbf{z} \in \mathbb{R}^{N_z}$ *is a solution to*

$$\begin{cases} \mathbb{A}\mathbf{z} = \Pi M_{\mathbf{zz}}^{-1}\mathbf{F}, \\ \mathbf{z} \in \mathbb{H}_h, \end{cases} \tag{3.18}$$

then $(\mathbf{z}, \boldsymbol{\eta}) \in \mathbb{R}^{N_z} \times \mathbb{R}^{N_\eta}$ *is a solution to*

$$\begin{cases} A_{\mathbf{zz}}\mathbf{z} + A_{\mathbf{z}\boldsymbol{\eta}}\boldsymbol{\eta} = \mathbf{F}, \\ A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z} = 0, \end{cases} \tag{3.19}$$

where $\boldsymbol{\eta} = (A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}})^{-1} A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1}(\mathbf{F} - A_{\mathbf{zz}}\mathbf{z})$.

Conversely, if $(\mathbf{z}, \boldsymbol{\eta}) \in \mathbb{R}^{N_z} \times \mathbb{R}^{N_\eta}$ *is a solution to* (3.19)*, then* $\mathbf{z} \in \mathbb{R}^{N_z}$ *is a solution to* (3.18)*.*

Proof. The reader can adapt the proof of Lemma 3.3.5. □

3.3.2 The projected adjoint system

The adjoint problem of (3.13) with respect to the natural scalar product of $\mathbb{R}^{N_{\text{tot}}}$ defined by $(U, V) \mapsto U^T V$ is

$$\begin{cases} -M_{\mathbf{zz}}\dfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi} = A_{\mathbf{zz}}^T\boldsymbol{\Phi} + A_{\boldsymbol{\eta}\mathbf{z}}^T\boldsymbol{\zeta}, \\ A_{\mathbf{z}\boldsymbol{\eta}}^T\boldsymbol{\Phi} = 0. \end{cases} \tag{3.20}$$

Similarly as in the previous section, the goal is to rewrite the system (3.20) as a system satisfied by $\boldsymbol{\Phi}$ in which the multiplier has been eliminated.

We define the space

$$\widetilde{\mathbb{H}_h} = \mathrm{Ker}(A_{\mathbf{z}\boldsymbol{\eta}}^T), \tag{3.21}$$

which is the space associated to the adjoint problem. We also define the following projectors.

Lemma 3.3.7. *The projector* $\widetilde{\Pi}$ *of* $\mathbb{R}^{N_z}$ *onto* $\widetilde{\mathbb{H}_h}$ *parallel to* $\mathrm{Im}(M_{\mathbf{zz}}^{-1}A_{\boldsymbol{\eta}\mathbf{z}}^T)$ *is defined by*

$$\widetilde{\Pi} = \mathrm{I} - M_{\mathbf{zz}}^{-1}A_{\boldsymbol{\eta}\mathbf{z}}^T(A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1} A_{\boldsymbol{\eta}\mathbf{z}}^T)^{-1} A_{\mathbf{z}\boldsymbol{\eta}}^T.$$

In the same way, the projector $\widetilde{\Pi}^T$ *of* $\mathbb{R}^{N_z}$ *onto* $\mathrm{Ker}(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1})$ *parallel to* $\mathrm{Im}(A_{\mathbf{z}\boldsymbol{\eta}})$ *is defined by*

$$\widetilde{\Pi}^T = \mathrm{I} - A_{\mathbf{z}\boldsymbol{\eta}}(A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}})^{-1} A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1}.$$

Moreover, we have the following relations

$$\Pi M_{\mathbf{zz}}^{-1} = M_{\mathbf{zz}}^{-1}\widetilde{\Pi}^T, \qquad M_{\mathbf{zz}}^{-1}\Pi^T = \widetilde{\Pi}M_{\mathbf{zz}}^{-1}, \qquad M_{\mathbf{zz}}\widetilde{\Pi} = \Pi^T M_{\mathbf{zz}}, \qquad M_{\mathbf{zz}}\Pi = \widetilde{\Pi}^T M_{\mathbf{zz}}.$$

Proof. This proof is a direct adaptation of the proof of Lemma 3.3.2. For the proof of the relations, we use a direct computation with the expressions of the projectors. □

Remark 3.3.8. In our case of study, because $A_{\mathbf{z}\boldsymbol{\eta}}^T = A_{\boldsymbol{\eta}\mathbf{z}}$, we have $\Pi = \widetilde{\Pi}$, $\Pi^T = \widetilde{\Pi}^T$ and $\mathbb{H}_h = \widetilde{\mathbb{H}_h}$. However, we do not use these relations since we want to treat the more general case in which they are not valid.

We can show the following result

Lemma 3.3.9. *The system* (3.20) *is equivalent to*

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{\Pi}\boldsymbol{\Phi} = \mathbb{A}^{\#}\widetilde{\Pi}\boldsymbol{\Phi}, \\ (\mathrm{I}-\widetilde{\Pi})\boldsymbol{\Phi} = 0, \\ \boldsymbol{\zeta} = -(A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1} A_{\boldsymbol{\eta}\mathbf{z}}^T)^{-1} A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1} A_{\mathbf{zz}}^T \boldsymbol{\Phi}, \end{cases} \tag{3.22}$$

where $\mathbb{A}^{\#} = \widetilde{\Pi} M_{\mathbf{zz}}^{-1} A_{\mathbf{zz}}^T$.

Proof. This proof is a straightforward adaptation of the one of Lemma 3.3.5. □

Remark 3.3.10. We have the relations $(\mathrm{I}-\Pi)^T M_{\mathbf{zz}} = M_{\mathbf{zz}}(\mathrm{I}-\widetilde{\Pi})$ and $(\mathbb{A}\Pi)^T M_{\mathbf{zz}} = M_{\mathbf{zz}}\mathbb{A}^{\#}\widetilde{\Pi}$. In that sense, the problem (3.22) is the adjoint problem of (3.16) with respect to the scalar product of $\mathbb{R}^{N_z}$ defined by $(U,V) \mapsto U^T M_{\mathbf{zz}} V$ which is the discretization of the scalar product $(.,.)_0$ defined in (3.7). The problem (3.22) is then the discretized version of the adjoint problem considered in the continuous case.

The following lemma is useful in Section 3.4.

Lemma 3.3.11. *If* $\boldsymbol{\Phi} \in \mathbb{R}^{N_z}$ *is a solution to*

$$\begin{cases} \mathbb{A}^{\#}\boldsymbol{\Phi} = \widetilde{\Pi} M_{\mathbf{zz}}^{-1}\mathbf{F}, \\ \boldsymbol{\Phi} \in \widetilde{\mathbb{H}_h}, \end{cases} \tag{3.23}$$

then $(\boldsymbol{\Phi}, \boldsymbol{\zeta}) \in \mathbb{R}^{N_z} \times \mathbb{R}^{N_\eta}$ *is a solution to*

$$\begin{cases} A_{\mathbf{zz}}^T\boldsymbol{\Phi} + A_{\boldsymbol{\eta}\mathbf{z}}^T\boldsymbol{\zeta} = \mathbf{F}, \\ A_{\mathbf{z}\boldsymbol{\eta}}^T\boldsymbol{\Phi} = 0, \end{cases} \tag{3.24}$$

where $\boldsymbol{\zeta} = (A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1} A_{\boldsymbol{\eta}\mathbf{z}}^T)^{-1} A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1}(\mathbf{F} - A_{\mathbf{zz}}^T\boldsymbol{\Phi})$.

Conversely, if $(\boldsymbol{\Phi}, \boldsymbol{\zeta}) \in \mathbb{R}^{N_z} \times \mathbb{R}^{N_\eta}$ *is a solution to* (3.24)*, then* $\boldsymbol{\Phi} \in \mathbb{R}^{N_z}$ *is a solution to* (3.23)*.*

3.3.3 Spectral decomposition of the operators

We have already noticed in Section 3.3.1 that $A_{\mathbf{z}\boldsymbol{\eta}}^T$ and $A_{\boldsymbol{\eta}\mathbf{z}}$ are full ranked. Hence their kernels have the same dimension. We denote $N_\pi = \dim(\mathbb{H}_h) = \dim(\widetilde{\mathbb{H}_h})$.

In what follows, we are looking for a decomposition of $\mathbb{H}_h$ (respectively $\widetilde{\mathbb{H}_h}$) into a sum of the generalized eigenspaces of the operator $\mathbb{A}$ (respectively $\mathbb{A}^{\#}$).

We say that $\mathbf{f} \in \mathbb{C}^{N_z}\backslash\{0\}$ is an eigenvector of $\mathbb{A}$ in $\mathbb{H}_h$ associated to the eigenvalue $\beta \in \mathbb{C}$ if $(\beta, \mathbf{f})$ is a solution to

$$\begin{cases} \mathbb{A}\mathbf{f} = \beta\mathbf{f}, \\ \mathbf{f} \in \mathbb{H}_h. \end{cases} \tag{3.25}$$

We say that $\mathbf{f} \in \mathbb{C}^{N_z}\backslash\{0\}$ is an eigenvector of $\mathbb{A}^{\#}$ in $\widetilde{\mathbb{H}_h}$ associated to the eigenvalue $\beta \in \mathbb{C}$ if $(\beta, \mathbf{f})$ is a solution to

$$\begin{cases} \mathbb{A}^{\#}\mathbf{f} = \beta\mathbf{f}, \\ \mathbf{f} \in \widetilde{\mathbb{H}_h}. \end{cases} \tag{3.26}$$

We say that a vector $\mathbf{f}^k \in \mathbb{C}^{N_z} \backslash \{0\}$ is a generalized eigenvector of order k for the problem (3.25) associated with a solution $(\beta, \mathbf{f}) \in \mathbb{C} \times \mathbb{H}_h$ of (3.25) if there exists $k \in \mathbb{N}^*$ such that

$$\begin{cases} (\mathbb{A} - \beta \mathrm{I})^k \mathbf{f}^k = \mathbf{f}, \\ \mathbf{f} \in \mathbb{H}_h. \end{cases} \tag{3.27}$$

In the same way, we say that a vector $\mathbf{f}^k \in \mathbb{C}^{N_z} \backslash \{0\}$ is a generalized eigenvector of order k for the problem (3.26) associated with a solution $(\beta, \mathbf{f}) \in \mathbb{C} \times \widetilde{\mathbb{H}_h}$ of (3.26) if there exists $k \in \mathbb{N}^*$ such that

$$\begin{cases} (\mathbb{A}^{\#} - \beta \mathrm{I})^k \mathbf{f}^k = \mathbf{f}, \\ \mathbf{f} \in \widetilde{\mathbb{H}_h}. \end{cases} \tag{3.28}$$

We now build a family of eigenvectors and generalized eigenvectors of $\mathbb{A}$ (respectively $\mathbb{A}^{\#}$).

Theorem 3.3.12. *There exist two matrices $\Psi \in \mathbb{C}^{N_z \times N_\pi}$ and $\widetilde{\Psi} \in \mathbb{C}^{N_z \times N_\pi}$ such that*
- *The columns of Ψ are eigenvectors and generalized eigenvectors of $\mathbb{A}$ and form a basis of $\mathbb{H}_h$,*
- *the columns of $\widetilde{\Psi}$ are eigenvectors and generalized eigenvectors of $\mathbb{A}^{\#}$ and form a basis of $\widetilde{\mathbb{H}_h}$,*
- *we have the decompositions*

$$\Lambda_{\mathbb{C}} = \widetilde{\Psi}^T M_{\mathbf{zz}} \mathbb{A} \Psi \in \mathbb{C}^{N_\pi \times N_\pi} \qquad \text{and} \qquad \Lambda_{\mathbb{C}}^T = \Psi^T M_{\mathbf{zz}} \mathbb{A}^{\#} \widetilde{\Psi} \in \mathbb{C}^{N_\pi \times N_\pi},$$

where $\Lambda_{\mathbb{C}}$ is a decomposition of $\mathbb{A}$ into complex Jordan blocks.
- *We have the following biorthogonality condition*

$$\Psi^T M_{\mathbf{zz}} \widetilde{\Psi} = \mathrm{I}_{N_\pi}. \tag{3.29}$$

The following lemma will be proven at the same time as Theorem 3.3.12.

Lemma 3.3.13. *Let $\mathbf{g} \in \widetilde{\mathbb{H}_h}$, then $\Psi^T M_{\mathbf{zz}} \mathbf{g} = 0 \Rightarrow \mathbf{g} = 0$.*

Proof of Theorem 3.3.12 and Lemma 3.3.13.

•The columns of Ψ are generalized eigenvectors of $\mathbb{A}$ and form a basis of $\mathbb{H}_h$: We know that $\mathbb{A} \in \mathcal{L}(\mathbb{H}_h)$, then there exists a basis $\Psi = (\mathbf{\Psi}^i)_{1 \le i \le N_\pi}$ of $\mathbb{H}_h$ composed of eigenvectors and generalized eigenvectors of $\mathbb{A}$.

•The biorthogonality condition and the columns of $\widetilde{\Psi}$ form a basis of $\widetilde{\mathbb{H}_h}$: The matrix $M_{\mathbf{zz}} \in \mathbb{R}^{N_z \times N_z}$ is invertible hence there exist two basis $(\mathbf{x}_j)_{1 \le j \le N_z}$ and $(\tilde{\mathbf{x}}_k)_{1 \le k \le N_z}$ of $\mathbb{R}^{N_z}$ satisfying the biorthogonality condition

$$\forall j, k \in [1, N_z], \quad \mathbf{x}_j^T M_{\mathbf{zz}} \tilde{\mathbf{x}}_k = \delta_{jk}.$$

For every $i \in [1, N_\pi]$, we write the coordinates of $\mathbf{\Psi}^i$ in $(\mathbf{x}_j)_{1 \le j \le N_z}$:

$$\forall i \in [1, N_\pi], \qquad \mathbf{\Psi}^i = \sum_{1 \le j \le N_z} a_j^i \mathbf{x}_j.$$

We consider the matrix $A \in \mathbb{C}^{N_z \times N_\pi}$ whose coefficients are $A_{ij} = a_i^j$ for $1 \le i \le N_z$ and $1 \le j \le N_\pi$. The rank of this matrix is N_π.

The matrix A^TA is the Gram matrix of $(\mathbf{\Psi}^i)$ which is a basis of $\mathbb{H}_h$, it is then invertible. We consider $A^\dagger = (A^TA)^{-1}A^T \in \mathbb{C}^{N_\pi \times N_z}$ the pseudo–inverse of A, it is of rank N_π. We denote $b_j^i = A^\dagger_{ji}$ for $1 \leq j \leq N_\pi$ and $1 \leq i \leq N_z$ and we define

$$\text{for every } i \in [1, N_\pi], \quad \widetilde{\mathbf{\Psi}}^i = \widetilde{\Pi} \sum_{j=1}^{N_z} b_i^j \tilde{\mathbf{x}}_j.$$

Then $\widetilde{\mathbf{\Psi}}^i$ belongs to $\widetilde{\mathbb{H}_h} = \mathrm{Im}(\widetilde{\Pi})$ and

$$\begin{aligned}(\mathbf{\Psi}^i)^T M_{\mathbf{zz}} \widetilde{\mathbf{\Psi}}^k &= \sum_{\ell=1}^{N_z} b_k^\ell (\mathbf{\Psi}^i)^T M_{\mathbf{zz}} \widetilde{\Pi} \tilde{\mathbf{x}}_\ell \ = \sum_{\ell=1}^{N_z} b_k^\ell (\mathbf{\Psi}^i)^T \Pi^T M_{\mathbf{zz}} \tilde{\mathbf{x}}_\ell = \sum_{\ell=1}^{N_z} b_k^\ell (\Pi \mathbf{\Psi}^i)^T M_{\mathbf{zz}} \tilde{\mathbf{x}}_\ell \\ &= \sum_{j,\ell=1}^{N_z} a_j^i b_k^\ell (\mathbf{x}_j)^T M_{\mathbf{zz}} \tilde{\mathbf{x}}_\ell = \sum_{j,\ell=1}^{N_z} a_j^i b_k^\ell \delta_{j\ell} = \sum_{j=1}^{N_z} a_j^i b_k^j = \delta_{ik},\end{aligned}$$

since $A^\dagger A = \mathrm{I}_{N_\pi}$.

We now prove that $\widetilde{\Psi} = (\widetilde{\mathbf{\Psi}}^i)$ is a basis of $\widetilde{\mathbb{H}_h} = \mathrm{Im}(\widetilde{\Pi})$. Let $(c_k)_{1 \leq k \leq N_\pi}$ such that $\sum_{k=1}^{N_\pi} c_k \widetilde{\mathbf{\Psi}}^k = 0$. Then, for every j, $0 = \sum_{k=1}^{N_\pi} c_k (\mathbf{\Psi}^j)^T M_{\mathbf{zz}} \widetilde{\mathbf{\Psi}}^k = c_j$. The family $\widetilde{\Psi}$ is then a basis of $\widetilde{\mathbb{H}_h} = \mathrm{Im}(\widetilde{\Pi})$.

•<u>The decomposition of $\mathbb{A}$:</u> If $\mathbf{\Psi}^i$ is an eigenvector of $\mathbb{A}$, then we have $\mathbb{A}\mathbf{\Psi}^i = \lambda \mathbf{\Psi}^i$. Let $(\mathbf{\Psi}^{i+1}, ..., \mathbf{\Psi}^{i+k})$ be the generalized eigenvectors associated to $(\lambda, \mathbf{\Psi}^i)$ such that $(\mathbb{A} - \lambda \mathrm{I})\mathbf{\Psi}^{\ell+1} = \mathbf{\Psi}^\ell$, then $\mathbb{A}\mathbf{\Psi}^{\ell+1} = \lambda \mathbf{\Psi}^{\ell+1} + \mathbf{\Psi}^\ell$.

Hence we have

$$\mathbb{A}\Psi = \Psi \Lambda_{\mathbb{C}},$$

where $\Lambda_{\mathbb{C}}$ is a Jordan block matrix. Now, with the biorthogonality relation (3.29), we have $\widetilde{\Psi}^T M_{\mathbf{zz}} \Psi = \mathrm{I}_{N_\pi}$ and

$$\Lambda_{\mathbb{C}} = \widetilde{\Psi}^T M_{\mathbf{zz}} \mathbb{A} \Psi.$$

•<u>The decomposition of $\mathbb{A}^\#$:</u> We start from $\Lambda_{\mathbb{C}} = \widetilde{\Psi}^T M_{\mathbf{zz}} \mathbb{A}\Psi$, we have

$$\Lambda_{\mathbb{C}}^T = \Psi^T \mathbb{A}^T M_{\mathbf{zz}} \widetilde{\Psi} = \Psi^T A_{\mathbf{zz}}^T M_{\mathbf{zz}}^{-1} \Pi^T M_{\mathbf{zz}} \widetilde{\Psi} = \Psi^T A_{\mathbf{zz}}^T \widetilde{\Pi} \widetilde{\Psi},$$

Moreover, Ψ is a basis of $\mathbb{H}_h = \mathrm{Im}(\Pi)$ and $\widetilde{\Psi}$ is a basis of $\widetilde{\mathbb{H}_h} = \mathrm{Im}(\widetilde{\Pi})$. Then, $\Pi\Psi = \Psi$, $\widetilde{\Pi}\widetilde{\Psi} = \widetilde{\Psi}$ and

$$\Lambda_{\mathbb{C}}^T = \Psi^T \Pi^T A_{\mathbf{zz}}^T \widetilde{\Psi} = \Psi^T M_{\mathbf{zz}} \widetilde{\Pi} M_{\mathbf{zz}}^{-1} A_{\mathbf{zz}}^T \widetilde{\Psi} = \Psi^T M_{\mathbf{zz}} \mathbb{A}^\# \widetilde{\Psi}.$$

•<u>Proof of Lemma 3.3.13</u> Let $\mathbf{g} \in \widetilde{\mathbb{H}_h}$, then there exists $(a_k)_{1 \leq k \leq N_\pi}$ such that $\mathbf{g} = \sum_{k=1}^{N_\pi} a_k \widetilde{\mathbf{\Psi}}^k$. Moreover, the biorthogonality condition (3.29) can be rewriten as $\Psi^T M_{\mathbf{zz}} \widetilde{\mathbf{\Psi}}^k = e_k$ where $(e_k)_j = \delta_{kj}$ (for $1 \leq k \leq N_\pi$ and $1 \leq j \leq N_z$). Then, $\Psi^T M_{\mathbf{zz}} \mathbf{g} = 0$ implies that $a_k = 0$ for every k and finally $\mathbf{g} = 0$.

•<u>The columns of $\widetilde{\Psi}$ are eigenvectors and generalized eigenvectors of $\mathbb{A}^\#$:</u> The last identity can be rewriten as

$$\Psi^T M_{\mathbf{zz}} (\mathbb{A}^\# \widetilde{\Psi} - \widetilde{\Psi} \Lambda_{\mathbb{C}}^T) = 0.$$

Lemma 3.3.13 implies that

$$\mathbb{A}^\# \widetilde{\Psi} = \widetilde{\Psi} \Lambda_{\mathbb{C}}^T.$$

Then, the columns of $\widetilde{\Psi}$ are eigenvectors and generalized eigenvectors of $\mathbb{A}^\#$. The proof is complete. □

The matrices Ψ and $\widetilde{\Psi}$ a priori contain complex valued columns. We now build matrices containing only real valued columns.

Theorem 3.3.14. *There exist two matrices $E \in \mathbb{R}^{N_z \times N_\pi}$ and $\widetilde{E} \in \mathbb{R}^{N_z \times N_\pi}$ such that*

— *the columns of E form a basis of $\mathbb{H}_h$,*

— *the columns of $\widetilde{E}$ form a basis of $\widetilde{\mathbb{H}_h}$,*

— *we have the decompositions*

$$\Lambda_{\mathbb{R}} = \widetilde{E}^T M_{\mathbf{zz}} \mathbb{A} E \qquad \textit{and} \qquad \Lambda_{\mathbb{R}}^T = E^T M_{\mathbf{zz}} \mathbb{A}^{\#} \widetilde{E},$$

where $\Lambda_{\mathbb{R}} \in \mathbb{R}^{N_\pi \times N_\pi}$ is a real Jordan matrix (see for instance [96, p.16]),

— *we have the biorthogonality condition*

$$E^T M_{\mathbf{zz}} \widetilde{E} = \mathrm{I}_{N_\pi}.$$

Proof. We denote $(\beta_k)_{1 \le k \le N_\pi}$ the eigenvalues of $\mathbb{A}$ which are counted as many times as their multiplicity. We denote $\boldsymbol{\Psi}^k$ (resp. $\widetilde{\boldsymbol{\Psi}}^k$) the k^{th} column of Ψ (resp. $\widetilde{\Psi}$) associated to β_k (resp. to $\overline{\beta_k}$). For each $1 \le k \le N_\pi$,

• if β_k is real, then we can consider that $\boldsymbol{\Psi}^k$ and $\widetilde{\boldsymbol{\Psi}}^k$ are also real. We then set $\boldsymbol{E}^k = \boldsymbol{\Psi}^k$ and $\widetilde{\boldsymbol{E}}^k = \widetilde{\boldsymbol{\Psi}}^k$.

• If β_k is not real, as $\mathbb{A}$ is a real matrix, there exists $j \neq k$ such that $\beta_k = \overline{\beta_j}$ and these eigenvalues have the same multiplicity. We may assume that $\boldsymbol{\Psi}^k = \overline{\boldsymbol{\Psi}^j}$ and $\widetilde{\boldsymbol{\Psi}}^k = \overline{\widetilde{\boldsymbol{\Psi}}^j}$. We then set $\boldsymbol{E}^k = \sqrt{2}\mathcal{R}e(\boldsymbol{\Psi}^k)$, $\widetilde{\boldsymbol{E}}^k = \sqrt{2}\mathcal{R}e(\widetilde{\boldsymbol{\Psi}}^k)$, $\boldsymbol{E}^j = \sqrt{2}\mathcal{I}m(\boldsymbol{\Psi}^k)$ and $\widetilde{\boldsymbol{E}}^j = \sqrt{2}\mathcal{I}m(\widetilde{\boldsymbol{\Psi}}^k)$.

We denote E (resp. $\widetilde{E}$) the matrix whose k^{th} column is $\boldsymbol{E}^k$ (resp. $\widetilde{\boldsymbol{E}}^k$) for $1 \le k \le N_\pi$. The properties of Ψ and $\widetilde{\Psi}$ which are stated in Theorem 3.3.12 imply the properties of E and $\widetilde{E}$. ☐

3.3.4 The projected systems

We denote $(\beta_j)_{1 \le j \le N_\pi}$ the eigenvalues of the matrix $\mathbb{A}$. We denote by $\mathrm{G}_{\mathbb{R}}(\beta_j)$ the real generalized eigenspace of $\mathbb{A}$ (obtained by considering real and imaginary parts of the complex eigenvectors of $\mathbb{A}$) and by $\mathrm{G}_{\mathbb{R}}^*(\beta_j)$ the real generalized eigenspace of $\mathbb{A}^{\#}$ associated to the eigenvalue β_j.

Let $\delta > 0$, we define J_u as the finite subset of $\mathbb{N}$ such that

$$\forall j \in J_u \Leftrightarrow -\delta \le \mathcal{R}e(\beta_j). \tag{3.30}$$

We define the following spaces

$$\mathbb{Z}_u = \bigoplus_{j \in J_u} \mathrm{G}_{\mathbb{R}}(\beta_j) \quad \text{and} \quad \mathbb{Z}_u^* = \bigoplus_{j \in J_u} \mathrm{G}_{\mathbb{R}}^*(\beta_j),$$

and $d_u = \dim(\mathbb{Z}_u) = \dim(\mathbb{Z}_u^*)$.

The set J_u and the families $(\boldsymbol{E}^i)_{1 \le i \le N_\pi}$ and $(\widetilde{\boldsymbol{E}}^i)_{1 \le i \le N_\pi}$ can be chosen such that

$$\mathbb{Z}_u = \mathrm{Vect}_{\mathbb{R}}((\boldsymbol{E}^i)_{1 \le i \le d_u}) \quad \text{and} \quad \mathbb{Z}_u^* = \mathrm{Vect}_{\mathbb{R}}((\widetilde{\boldsymbol{E}}^i)_{1 \le i \le d_u}).$$

We also define

$$\mathbb{Z}_s = \mathrm{Vect}_{\mathbb{R}}((\boldsymbol{E}^i)_{d_u+1 \le i \le N_\pi}) \quad \text{and} \quad \mathbb{Z}_s^* = \mathrm{Vect}_{\mathbb{R}}((\widetilde{\boldsymbol{E}}^i)_{d_u+1 \le i \le N_\pi}),$$

and $d_s = \dim(\mathbb{Z}_s) = \dim(\mathbb{Z}_s^*)$.

We have

$$\mathbb{H}_h = \mathbb{Z}_u \bigoplus \mathbb{Z}_s \quad \text{and} \quad \widetilde{\mathbb{H}_h} = \mathbb{Z}_u^* \bigoplus \mathbb{Z}_s^*.$$

We define the following matrices :

— $E_u \in \mathbb{R}^{N_z \times d_u}$ is the matrix whose columns are $(\boldsymbol{E}^i)_{1 \leq i \leq d_u}$,

— $E_s \in \mathbb{R}^{N_z \times d_s}$ is the matrix whose columns are $(\boldsymbol{E}^i)_{d_u+1 \leq i \leq N_\pi}$,

— $\widetilde{E}_u \in \mathbb{R}^{N_z \times d_u}$ is the matrix whose columns are $(\widetilde{\boldsymbol{E}}^i)_{1 \leq i \leq d_u}$,

— $\widetilde{E}_s \in \mathbb{R}^{N_z \times d_s}$ is the matrix whose columns are $(\widetilde{\boldsymbol{E}}^i)_{d_u+1 \leq i \leq N_\pi}$,

— $\Lambda_u \in \mathbb{R}^{d_u \times d_u}$ and $\Lambda_s \in \mathbb{R}^{d_s \times d_s}$ are the matrices such that $\Lambda_{\mathbb{R}} = \begin{pmatrix} \Lambda_u & 0_{\mathbb{R}^{d_u \times d_s}} \\ 0_{\mathbb{R}^{d_s \times d_u}} & \Lambda_s \end{pmatrix}$.

Lemma 3.3.15. *We have the following relations*

$$\begin{array}{ll} E = [E_u \ E_s], & \widetilde{E} = [\widetilde{E}_u \ \widetilde{E}_s], \\ E_u^T M_{\mathbf{zz}} \widetilde{E}_u = \mathrm{I}_{d_u}, & E_u^T M_{\mathbf{zz}} \widetilde{E}_s = 0_{d_u \times d_s}, \\ E_s^T M_{\mathbf{zz}} \widetilde{E}_u = 0_{d_s \times d_u}, & E_s^T M_{\mathbf{zz}} \widetilde{E}_s = \mathrm{I}_{d_s}, \\ \widetilde{E}_u^T M_{\mathbf{zz}} \mathbb{A} E_u = \Lambda_u, & \widetilde{E}_u^T M_{\mathbf{zz}} \mathbb{A} E_s = 0_{d_u \times d_s}, \\ \widetilde{E}_s^T M_{\mathbf{zz}} \mathbb{A} E_u = 0_{d_s \times d_u}, & \widetilde{E}_s^T M_{\mathbf{zz}} \mathbb{A} E_s = \Lambda_s. \end{array}$$

Proof. These identities are consequences of the relations

$$\begin{pmatrix} E_u^T \\ E_s^T \end{pmatrix} M_{\mathbf{zz}} (\widetilde{E}_u \ \ \widetilde{E}_s) = \begin{pmatrix} \mathrm{I}_{d_u} & 0_{d_u \times d_s} \\ 0_{d_s \times d_u} & \mathrm{I}_{d_s} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{E}_u^T \\ \widetilde{E}_s^T \end{pmatrix} M_{\mathbf{zz}} \mathbb{A} (E_u \ \ E_s) = \begin{pmatrix} \Lambda_u & 0 \\ 0 & \Lambda_s \end{pmatrix}.$$

□

We define $\Pi_u \in \mathbb{R}^{N_z \times N_z}$ (respectively $\Pi_s \in \mathbb{R}^{N_z \times N_z}$) the projector of $\mathbb{R}^{N_z}$ onto $\mathbb{Z}_u$ (respectively $\mathbb{Z}_s$) parallel to $\mathbb{Z}_s \bigoplus \mathrm{Ker}(\Pi)$ (respectively $\mathbb{Z}_u \bigoplus \mathrm{Ker}(\Pi)$).

Lemma 3.3.16. *We have the following identities*

$$\Pi_u = E_u \widetilde{E}_u^T M_{\mathbf{zz}}, \quad \Pi_s = E_s \widetilde{E}_s^T M_{\mathbf{zz}}, \quad \Pi = E_u \widetilde{E}_u^T M_{\mathbf{zz}} + E_s \widetilde{E}_s^T M_{\mathbf{zz}}.$$

Proof. • Let $\Pi^\# = E_u \widetilde{E}_u^T M_{\mathbf{zz}}$. With the help of Lemma 3.3.15, we compute $(\Pi^\#)^2 = \Pi^\#$. Hence $\Pi^\#$ is a projection of $\mathbb{R}^{N_z}$. We now compute its kernel and image spaces.

• Let $\mathbf{z} \in \mathbb{Z}_u$. The family E_u is a basis of $\mathbb{Z}_u$, then there exists $\mathbf{z}_u \in \mathbb{R}^{d_u}$ such that $\mathbf{z} = E_u \mathbf{z}_u$. The vector $\mathbf{z}_u$ represents the coordinates of $\mathbf{z}$ in the basis E_u. By using the relations of Lemma 3.3.15, we prove that $\Pi^\# \mathbf{z} = E_u \widetilde{E}_u^T M_{\mathbf{zz}} E_u \mathbf{z}_u = E_u \mathbf{z}_u = \mathbf{z}$. Hence $\mathrm{Im}(\Pi_u) \subset \mathrm{Im}(\Pi^\#)$.

• Let $\mathbf{z} \in \mathrm{Ker}(\Pi_u) = \mathrm{Im}(M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}}) \bigoplus \mathbb{Z}_s$, then there exist $\mathbf{z}^\# \in \mathbb{R}^{N_z}$ and $\mathbf{z}_s \in \mathbb{R}^{d_s}$ such that $\mathbf{z} = M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}} \mathbf{z}^\# + E_s \mathbf{z}_s$. We compute $\Pi^\# \mathbf{z} = E_u \widetilde{E}_u^T M_{\mathbf{zz}} (M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}} \mathbf{z}^\# + E_s \mathbf{z}_s) = E_u \widetilde{E}_u^T A_{\mathbf{z}\boldsymbol{\eta}} \mathbf{z}^\# = 0$ as $A_{\mathbf{z}\boldsymbol{\eta}}^T \widetilde{E}_u = 0$. Then $\mathrm{Ker}(\Pi_u) \subset \mathrm{Ker}(\Pi^\#)$.

• The dimensions of the spaces imply that $\Pi^\# = \Pi_u$.

• The same strategy can be used to prove the relation on Π_s, the relation on Π is a consequence of $\Pi = \Pi_u + \Pi_s$. This ends the proof. □

We have the following proposition.

Proposition 3.3.17. *The variable* $\mathbf{z} \in \mathrm{H}^1(0,T;\mathbb{H}_h)$ *fulfils*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}(t) = \mathbb{A}\mathbf{z}(t) + \mathbb{B}\mathbf{h}(t), \quad t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{3.31}$$

if and only if $\mathbf{z}_u = \Pi_u \mathbf{z}$ *and* $\mathbf{z}_s = \Pi_s \mathbf{z}$ *satisfy respectively*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_u(t) = \mathbb{A}_u\mathbf{z}_u(t) + \mathbb{B}_u\mathbf{h}(t), \quad t > 0, \\ \mathbf{z}_u(0) = \Pi_u\mathbf{z}_0, \end{cases} \tag{3.32}$$

and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_s(t) = \mathbb{A}_s\mathbf{z}_s(t) + \mathbb{B}_s\mathbf{h}(t), \quad t > 0, \\ \mathbf{z}_s(0) = \Pi_s\mathbf{z}_0, \end{cases} \tag{3.33}$$

where $\quad \mathbb{A}_u = \Pi_u M_{\mathbf{zz}}^{-1} A_{\mathbf{zz}}, \quad \mathbb{A}_s = \Pi_s M_{\mathbf{zz}}^{-1} A_{\mathbf{zz}}, \quad \mathbb{B}_u = \Pi_u M_{\mathbf{zz}}^{-1} B$ *and* $\mathbb{B}_s = \Pi_s M_{\mathbf{zz}}^{-1} B. \quad (3.34)$

Proof. • Let $\mathbf{z}$ fulfil (3.31). We use the projection Π_u and get

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}_u(t) = \mathbb{A}_u(\Pi_u\mathbf{z}(t) + \Pi_s\mathbf{z}(t)) + \mathbb{B}_u\mathbf{h}(t), \quad t > 0, \\ \mathbf{z}_u(0) = \Pi_u\mathbf{z}_0. \end{cases}$$

Moreover, $\mathbb{A}_u\Pi_s\mathbf{z} = E_u\widetilde{E}_u^T M_{\mathbf{zz}}\mathbb{A}E_s\widetilde{E}_s^T M_{\mathbf{zz}}\mathbf{z}$, and according to Lemma 3.3.15, we have $\widetilde{E}_u^T M_{\mathbf{zz}}\mathbb{A}E_s = 0$. Then, $\mathbb{A}_u\Pi_s\mathbf{z} = 0$ and we get (3.32).

We similarly get (3.33).

• Now, let $\mathbf{z}_u$ and $\mathbf{z}_s$ fulfilling (3.32) and (3.33). By combining the two systems and using the fact that $\Pi_u\mathbb{A}\Pi_s = \Pi_s\mathbb{A}\Pi_u = 0$, we prove that $\mathbf{z} = \mathbf{z}_u + \mathbf{z}_s$ satisfies (3.31). □

3.3.5 Computation of the linear feedback

Let $\delta > 0$, we assume the following hypothesis to be valid.

Hypothesis $(\widetilde{\mathcal{H}})_\delta$. Every eigenvector $(\mathbf{z}, \boldsymbol{\eta}) \in \mathbb{R}^{N_z} \times \mathbb{R}^{N_\eta}$ of the adjoint problem, associated to an eigenvalue $\overline{\beta}$ such that $\mathcal{R}e(\overline{\beta}) \geq -\delta$, that belongs to the kernel of the adjoint of the control operator is null, i.e.

$$\begin{pmatrix} A_{\mathbf{zz}}^T & A_{\boldsymbol{\eta}\mathbf{z}}^T \\ A_{\mathbf{z}\boldsymbol{\eta}}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{pmatrix} = \overline{\beta} \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{pmatrix} \qquad \text{and} \qquad B^T\mathbf{z} = 0 \qquad \Longrightarrow \qquad \mathbf{z} = 0.$$

The hypothesis $(\widetilde{\mathcal{H}})_\delta$ corresponds to an Hautus test on the adjoint system of (3.32). Hence, under this assumption, (3.32) is stabilizable.

In order to build a stabilizing feedback for (3.32), let $\omega > \delta$, we consider $\mathbf{R}_u \in \mathcal{L}(\mathbb{Z}_u)$ a solution to the Riccati equation

$$\begin{cases} \mathbf{R}_u = \mathbf{R}_u^T \geq 0, \\ \mathbf{R}_u(\mathbb{A}_u + \omega\Pi_u) + (\mathbb{A}_u^T + \omega\Pi_u^T)\mathbf{R}_u - \mathbf{R}_u\mathbb{B}_u\mathbb{B}_u^T\mathbf{R}_u = 0. \end{cases} \tag{3.35}$$

The existence of such a solution is given by Lemma 3.4.4 that is proven in the sequel. The value of the parameter ω is chosen in Section 3.5.2, it tunes the intensity of the feedback control.

This solution is used to construct the feedback control law

$$\mathbf{h} = \mathbf{K}_{\delta,\omega}\mathbf{z}_u = -\mathbb{B}_u^T\mathbf{R}_u\mathbf{z}_u,$$

that stabilizes the problem (3.32) with the exponential decay rate δ.

Remark 3.3.18. Note that the feedback matrix $\mathbf{K}_{\delta,\omega}$ does not depend on the choice of the eigenvectors of (3.25) and (3.27).

We can prove that this feedback law also stabilizes the whole problem (3.13).

Theorem 3.3.19. *For any* $\mathbf{z}_0 \in \mathrm{Ker}(A_{\boldsymbol{\eta}\mathbf{z}})$*, the solution to*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{z}(t) = \mathbb{A}\mathbf{z}(t) + \mathbb{B}\mathbf{K}_{\delta,\omega}\Pi_u\mathbf{z}(t), \quad t > 0, \\ A_{\boldsymbol{\eta}\mathbf{z}}\mathbf{z}(t) = 0, \quad t > 0, \\ \mathbf{z}(0) = \mathbf{z}_0. \end{cases} \tag{3.36}$$

satisfies the estimate

$$\|\mathbf{z}(t)\| \leq C\|\mathbf{z}_0\|e^{-\delta t}, \quad \forall t > 0.$$

Proof. According to Proposition 3.3.17, $\Pi_u\mathbf{z}$ satisfies (3.32) with $\mathbf{h} = \mathbf{K}_{\delta,\omega}\Pi_u\mathbf{z}$. Hence, an adaptation of [24, Section I.1, Theorem 3.1, p. 34] proves that $\mathbb{A}_u + \delta\mathrm{I} + \mathbb{B}_u\mathbf{K}_{\delta,\omega}$ generates a stable semigroup, and then we have

$$\|\Pi_u\mathbf{z}(t)\| \leq C\|\mathbf{z}_0\|e^{-\delta t}.$$

Moreover, according to Proposition 3.3.17, $\Pi_s\mathbf{z}$ satisfies (3.33) with $\mathbf{h} = \mathbf{K}_{\delta,\omega}\Pi_u\mathbf{z}$. And we know that $\mathbb{A}_s + \delta\mathrm{I}$ generates a stable semigroup (see the definition of J_u in (3.30)). Then,

$$\|\Pi_s\mathbf{z}(t)\| \leq C\|\mathbf{z}_0\|e^{-\delta t} + C\|\mathbf{z}_0\|\|\Pi_u\mathbf{z}(t)\| \leq C\|\mathbf{z}_0\|e^{-\delta t}.$$

This ends the proof. □

3.4 Practical computation of the feedback matrix

The strategy developed in Section 3.3 cannot be run efficiently in practice mainly because the matrices $(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}$ and $(A_{\boldsymbol{\eta}\mathbf{z}}^T M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}}^T)^{-1}$ are heavy to compute and because the Riccati equation (3.35) is of size $N_z \times N_z$. In the present section, we show how to adapt the previous strategy in order to compute numerically the matrix $\mathbf{K}_{\delta,\omega}$ in an efficient manner.

3.4.1 Equivalence between the eigenvalue problems

In what follows we show a practical way of computing the basis of complex valued eigenvectors and generalized eigenvectors of $\mathbb{A}$ (respectively $\mathbb{A}^{\#}$).

Lemma 3.4.1. *If the couple* $(\beta, \mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z}$ *is a solution to* (3.25)*, then the triplet* $(\beta, \mathbf{f}, \boldsymbol{\eta}_\mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z} \times \mathbb{C}^{N_\eta}$ *is a solution to the problem*

$$\begin{pmatrix} A_{\mathbf{zz}} & A_{\mathbf{z}\boldsymbol{\eta}} \\ A_{\boldsymbol{\eta}\mathbf{z}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \boldsymbol{\eta}_\mathbf{f} \end{pmatrix} = \beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \boldsymbol{\eta}_\mathbf{f} \end{pmatrix}, \tag{3.37}$$

where $\boldsymbol{\eta}_\mathbf{f} = -(A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{z}\boldsymbol{\eta}})^{-1}A_{\boldsymbol{\eta}\mathbf{z}}M_{\mathbf{zz}}^{-1}A_{\mathbf{zz}}\mathbf{f}$.

Conversely, if the triplet $(\beta, \mathbf{f}, \boldsymbol{\eta}_\mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z} \times \mathbb{C}^{N_\eta}$ *is a solution to the problem* (3.37)*, then the couple* $(\beta, \mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z}$ *is a solution to* (3.25).

Proof. This lemma is a consequence of Lemma 3.3.6 with $\mathbf{F} = \beta M_{\mathbf{zz}}\mathbf{f}$ and the fact that, for $\mathbf{f} \in \mathbb{H}_h$, we have $A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1} M_{\mathbf{zz}}\mathbf{f} = 0$. $\square$

Lemma 3.4.2. *If the couple $(\beta, \mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z}$ is a solution to (3.26), then the triplet $(\beta, \mathbf{f}, \boldsymbol{\zeta}_\mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z} \times \mathbb{C}^{N_\eta}$ is a solution to the problem*

$$\begin{pmatrix} A_{\mathbf{zz}}^T & A_{\boldsymbol{\eta}\mathbf{z}}^T \\ A_{\mathbf{z}\boldsymbol{\eta}}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \boldsymbol{\zeta}_\mathbf{f} \end{pmatrix} = \beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \boldsymbol{\zeta}_\mathbf{f} \end{pmatrix}, \tag{3.38}$$

where $\boldsymbol{\zeta}_\mathbf{f} = -(A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1} A_{\boldsymbol{\eta}\mathbf{z}}^T)^{-1} A_{\mathbf{z}\boldsymbol{\eta}}^T M_{\mathbf{zz}}^{-1} A_{\mathbf{zz}}^T \mathbf{f}$.

Conversely, if the triplet $(\beta, \mathbf{f}, \boldsymbol{\zeta}_\mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z} \times \mathbb{C}^{N_\eta}$ is a solution to the problem (3.38), then the couple $(\beta, \mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z}$ is a solution to (3.26).

Proof. This is a straightforward adaptation of the proof of Lemma 3.4.1, where we use Lemma 3.3.11. $\square$

A similar result can be proven for the generalized eigenvectors. We state the following theorem.

Theorem 3.4.3. *Let $(\beta, \mathbf{f}, \boldsymbol{\eta}_\mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z}\backslash\{0\} \times \mathbb{C}^{N_\eta}$ be a solution to (3.37). A vector $\mathbf{f}^k \in \mathbb{C}^{N_z}\backslash\{0\}$ is solution to (3.27) associated to $(\beta, \mathbf{f})$ if and only if there exists $\boldsymbol{\delta}_\mathbf{f}^k \in \mathbb{C}^{N_\eta}$ such that $(\mathbf{f}^k, \boldsymbol{\delta}_\mathbf{f}^k)$ is solution to the problem*

$$\left(\begin{pmatrix} A_{\mathbf{zz}} & A_{\mathbf{z}\boldsymbol{\eta}} \\ A_{\boldsymbol{\eta}\mathbf{z}} & 0 \end{pmatrix} - \beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \right)^k \begin{pmatrix} \mathbf{f}^k \\ \boldsymbol{\delta}_\mathbf{f}^k \end{pmatrix} = \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \boldsymbol{\eta}_\mathbf{f} \end{pmatrix}.$$

Similarly, let $(\beta, \mathbf{f}, \boldsymbol{\zeta}_\mathbf{f}) \in \mathbb{C} \times \mathbb{C}^{N_z} \times \mathbb{C}^{N_\eta}$ be a solution to (3.38). A vector $\mathbf{f}^k \in \mathbb{C}^{N_z}\backslash\{0\}$ is solution to (3.28) associated to $(\beta, \mathbf{f})$ if and only if there exists $\boldsymbol{\mu}_\mathbf{f}^k \in \mathbb{C}^{N_\eta}$ such that $(\mathbf{f}^k, \boldsymbol{\mu}_\mathbf{f}^k)$ is solution to the problem

$$\left(\begin{pmatrix} A_{\mathbf{zz}}^T & A_{\boldsymbol{\eta}\mathbf{z}}^T \\ A_{\mathbf{z}\boldsymbol{\eta}}^T & 0 \end{pmatrix} - \beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \right)^k \begin{pmatrix} \mathbf{f}^k \\ \boldsymbol{\mu}_\mathbf{f}^k \end{pmatrix} = \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \boldsymbol{\zeta}_\mathbf{f} \end{pmatrix}.$$

Proof. We use Lemma 3.3.6 and an induction argument.

$$\left(\begin{pmatrix} A_{\mathbf{zz}} & A_{\mathbf{z}\boldsymbol{\eta}} \\ A_{\boldsymbol{\eta}\mathbf{z}} & 0 \end{pmatrix} - \beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \mathbf{f}^{k+1} \\ \boldsymbol{\delta}_\mathbf{f}^{k+1} \end{pmatrix} = \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f}^k \\ \boldsymbol{\delta}_\mathbf{f}^k \end{pmatrix}$$

is equivalent to

$$\mathbf{f}^{k+1} \in \mathrm{Ker}(A_{\mathbf{z}\boldsymbol{\eta}}), \qquad (\mathbb{A} - \beta \mathrm{I})\mathbf{f}^{k+1} = \mathbf{f}^k,$$

with $\boldsymbol{\delta}_\mathbf{f}^{k+1} = (A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1} A_{\mathbf{z}\boldsymbol{\eta}})^{-1} A_{\boldsymbol{\eta}\mathbf{z}} M_{\mathbf{zz}}^{-1} (M_{\mathbf{zz}}(\beta \mathbf{f}^{k+1} + \mathbf{f}^k) - A_{\mathbf{zz}}\mathbf{f}^{k+1})$.

The proof is similar for the adjoint problem. $\square$

3.4.2 A practical way to compute the feedback matrix

Another problem faced when computing numerically the feedback matrix is the high dimensionality of the Riccati equation (3.35). In this section, we show how it is possible to transform the problem to have only to solve a low dimensional Riccati equation to conclude.

We construct the linear feedback law in the following way. We consider the solution $\widetilde{\mathbf{R}_u}$ to the equation

$$\begin{cases} \widetilde{\mathbf{R}_u} \in \mathcal{L}(\mathbb{R}^{d_u}), \quad \widetilde{\mathbf{R}_u} = \widetilde{\mathbf{R}_u}^T \geq 0, \\ \widetilde{\mathbf{R}_u}(\Lambda_u + \omega \mathrm{I}_{\mathbb{R}^{d_u}}) + (\Lambda_u^T + \omega \mathrm{I}_{\mathbb{R}^{d_u}})\widetilde{\mathbf{R}_u} - \widetilde{\mathbf{R}_u}\widetilde{E}_u^T B B^T \widetilde{E}_u \widetilde{\mathbf{R}_u} = 0. \end{cases} \tag{3.39}$$

Existence and uniqueness of the solution to (3.39) is given by [94].

We can link the solution of (3.39) and a solution of (3.35).

Lemma 3.4.4. *If the matrix $\mathbf{R}_u \in \mathbb{R}^{N_z \times N_z}$ is a solution to (3.35), then $\widetilde{\mathbf{R}_u} = E_u^T \mathbf{R}_u E_u$ is the solution to (3.39).*

Conversely, if $\widetilde{\mathbf{R}_u} \in \mathbb{R}^{d_u \times d_u}$ is the solution to (3.39), then $\mathbf{R}_u = M_{\mathbf{zz}} \widetilde{E}_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}}$ is a solution to (3.35).

Proof.
• If $\mathbf{R}_u$ satisfies (3.35), then

$$E_u^T \mathbf{R}_u (\mathbb{A}_u + \omega \Pi_u) E_u + E_u^T (\mathbb{A}_u^T + \omega \Pi_u) \mathbf{R}_u E_u - E_u^T \mathbf{R}_u \mathbb{B}_u \mathbb{B}_u^T \mathbf{R}_u E_u = 0.$$

We use $\widetilde{E}_u^T M_{\mathbf{zz}} \mathbb{A} E_u = \Lambda_u$ (see Lemma 3.3.15) and $\Pi_u = E_u \widetilde{E}_u^T M_{\mathbf{zz}}$, we get $\mathbb{A}_u E_u = E_u \widetilde{E}_u^T M_{\mathbf{zz}} \mathbb{A} E_u = E_u \Lambda_u$. Moreover, $\mathbb{B}_u = E_u \widetilde{E}_u^T B$, then $\widetilde{\mathbf{R}_u} = E_u^T \mathbf{R}_u E_u$ is solution to (3.39).
• Let $\widetilde{\mathbf{R}_u}$ be the solution to (3.39). We multiply the equation on the left by $M_{\mathbf{zz}} \widetilde{E}_u$ and on the right by $\widetilde{E}_u^T M_{\mathbf{zz}}$, we get

$$\begin{aligned} M_{\mathbf{zz}} \widetilde{E}_u \widetilde{\mathbf{R}_u} (\Lambda_u + \omega \mathrm{I}_{\mathbb{R}^{d_u}}) \widetilde{E}_u^T M_{\mathbf{zz}} + M_{\mathbf{zz}} \widetilde{E}_u (\Lambda_u^T + \omega \mathrm{I}_{\mathbb{R}^{d_u}}) \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}} & \\ - M_{\mathbf{zz}} \widetilde{E}_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T B B^T \widetilde{E}_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}} &= 0. \end{aligned}$$

We use the identites $\mathrm{I}_{\mathbb{R}^{d_u}} = E_u^T M_{\mathbf{zz}} \widetilde{E}_u$ and $\widetilde{E}_u^T M_{\mathbf{zz}} \mathbb{A}_u E_u = \Lambda_u$, we get

$$\begin{aligned} M_{\mathbf{zz}} \widetilde{E}_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}} (\mathbb{A}_u E_u \widetilde{E}_u^T M_{\mathbf{zz}} + \omega E_u \widetilde{E}_u^T M_{\mathbf{zz}}) + M_{\mathbf{zz}} \widetilde{E}_u E_u^T (\mathbb{A}_u^T + \omega \mathrm{I}_{\mathbb{R}^{N_z}}) M_{\mathbf{zz}} \widetilde{E}_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}} & \\ - M_{\mathbf{zz}} \widetilde{E}_u \widetilde{\mathbf{R}_u} (\widetilde{E}_u^T M_{\mathbf{zz}} E_u) \widetilde{E}_u^T B B^T \widetilde{E}_u (\widetilde{E}_u^T M_{\mathbf{zz}} E_u) \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}} &= 0. \end{aligned}$$

We finally use the identities $\Pi_u = E_u \widetilde{E}_u^T M_{\mathbf{zz}}$, $\mathbb{A}_u \Pi_u = \mathbb{A}_u$ and $\mathbb{B}_u = \Pi_u B = E_u \widetilde{E}_u^T B$, this proves that $\mathbf{R}_u = M_{\mathbf{zz}} E_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}}$ fulfils (3.35). □

It is more convenient to solve the equation (3.39), which is of size $d_u \times d_u$, than the equation (3.35), which is of size $N_z \times N_z$. Then, the main consequence of Lemma 3.4.4 is that we will solve (3.39) instead of (3.35).

We have $\Pi_u = E_u \widetilde{E}_u^T M_{\mathbf{zz}}$ (see Lemma 3.3.16) and $\mathbb{B}_u = \Pi_u M_{\mathbf{zz}}^{-1} B = E_u \widetilde{E}_u^T B$, then

$$\mathbf{K}_{\delta,\omega} = -\mathbb{B}_u^T \mathbf{R}_u = -B^T \widetilde{E}_u E_u^T \mathbf{R}_u E_u \widetilde{E}_u^T M_{\mathbf{zz}}.$$

Now, Lemma 3.4.4 gives

$$\mathbf{K}_{\delta,\omega} = -\mathbb{B}_u^T \mathbf{R}_u = -B^T \widetilde{E}_u \widetilde{\mathbf{R}_u} \widetilde{E}_u^T M_{\mathbf{zz}}. \tag{3.40}$$

3.5 Numerical computations in the fixed domain

In this section, we define properly the functions $\mathbf{X}$ and $\mathbf{\Phi^S}$ that will be used in the numerical simulations. Moreover, we study the numerical computation of the eigenvalues and eigenvectors of the problem. Finally, we give some details about the computation of the feedback matrix $\mathbf{K}_{\delta,\omega}$ defined in (3.40).

3.5.1 The diffeomorphisms used

3.5.1.1 The diffeomorphism X

<u>Deformation considerations :</u> We consider that every fibre of matter stays normal to the mid–line of the structure in every configuration. Hence the deformation of the structure is given by the deformation of the mid–line.

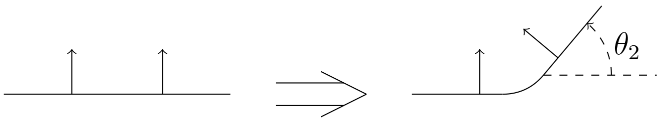


FIGURE 3.4 – The deformation of the mid–line.

<u>The deformation of the mid–line :</u> In the non–deformed configuration, the mid–line covers the interval $[0,1]$. Let $x_a < x_b$ be in $(0,1)$, we want the mid–line to be at rest in $(0, x_a)$ and be a straight line of slope θ_2 in $(x_b, 1)$. We have to choose carefully its behaviour in (x_a, x_b) in order to make it a $\mathscr{C}^1$ curve.

In order to have a smooth curve, we define it in the following way. Let $x_c = (x_a + x_b)/2$ and

$$f(x) = a(x - x_c)^2 + b(x - x_c) + c,$$

we want $y = f(x)$ to be a parabola passing through $(x_a, 0)$ and $(x_b, 0)$ with a tangent angle of respectively $-\tan(\theta_2/2)$ and $\tan(\theta_2/2)$ (see Fig. 3.5).

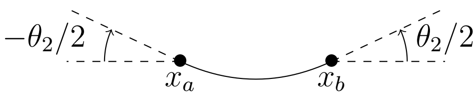


FIGURE 3.5 – Illustration of the function f.

We can compute the corresponding coefficients for f. We obtain the function

$$f(x) = \frac{\tan(\theta_2/2)}{x_b - x_a}(x - x_c)^2 - \tan(\theta_2/2)\frac{x_b - x_a}{4}.$$

The next step is to rotate this parabola around $(x_a, 0)$ of angle $\theta_2/2$, to extend it on the left hand-side by $y = 0$ and on the right by a straight line of slope θ_2. This will give the desired deformation for the mid–line (see Fig. 3.6).

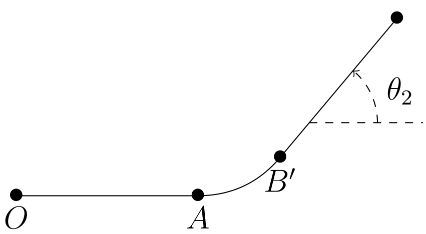


FIGURE 3.6 – The deformation of the mid–line.

The coordinates of the point B' in Fig. 3.6 are given by

$$\begin{cases} x_{B'} = x_a + (x_b - x_a)\cos(\theta_2/2), \\ y_{B'} = (x_b - x_a)\sin(\theta_2/2). \end{cases}$$

The expression of the mid–line is then

$$g_1(\ell) = \begin{cases} \ell & \text{if } \ell \le x_a, \\ x_a + (\ell - x_a)\cos(\theta_2/2) - f(\ell)\sin(\theta_2/2) & \text{if } \ell \in (x_a, x_b), \\ x_{B'} + (\ell - x_b)\cos\theta_2 & \text{if } \ell \ge x_b, \end{cases}$$

and

$$g_2(\ell) = \begin{cases} 0 & \text{if } \ell \le x_a, \\ (\ell - x_a)\sin(\theta_2/2) + f(\ell)\cos(\theta_2/2) & \text{if } \ell \in (x_a, x_b), \\ y_{B'} + (\ell - x_b)\sin\theta_2 & \text{if } \ell \ge x_b. \end{cases}$$

Deformation of the structure : The following vector

$$\mathbf{N}(\ell) = \begin{pmatrix} -g_1'(\ell) \\ g_2'(\ell) \end{pmatrix}$$

is normal to the mid–line. In the sequel, $\mathbf{y} = (y_1, y_2)$ denotes the lagrangian coordinates. We define the following diffeomorphism

$$\tilde{\mathbf{X}}(\theta_2, \mathbf{y}) = \begin{pmatrix} g_1(y_1) + y_2 \dfrac{N_1(y_1)}{|\mathbf{N}(y_1)|} \\ g_2(y_1) + y_2 \dfrac{N_2(y_1)}{|\mathbf{N}(y_1)|} \end{pmatrix}$$

which is represented in Fig. 3.7.

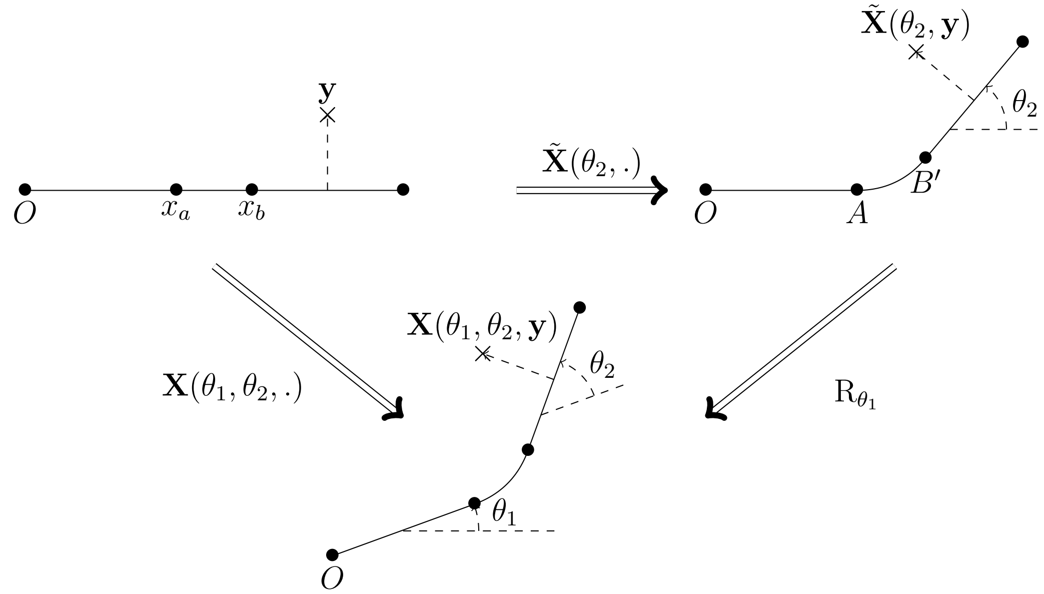


FIGURE 3.7 – The diffeomorphisms $\mathbf{X}$ and $\tilde{\mathbf{X}}$.

We divide by $|\mathbf{N}|$ to have a unitary vector. Note that $|\mathbf{N}(\ell)| = \sqrt{1 + (2a(\ell - x_c) + b)^2}$.

The rotation – the diffeomorphism $\mathbf{X}(\theta_1, \theta_2, .)$: We get the final diffeomorphism $\mathbf{X}(\theta_1, \theta_2, .)$ after a rotation of angle θ_1 around the center O,

$$\begin{aligned}
\mathbf{X}(\theta_1, \theta_2, \mathbf{y}) &= \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \tilde{\mathbf{X}}(\theta_2, \mathbf{y}) \\
&= \begin{pmatrix} \left(g_1(y_1) + y_2 \dfrac{N_1(y_1)}{|\mathbf{N}(y_1)|}\right)\cos\theta_1 - \left(g_2(y_1) + y_2 \dfrac{N_2(y_1)}{|\mathbf{N}(y_1)|}\right)\sin\theta_1 \\ \left(g_1(y_1) + y_2 \dfrac{N_1(y_1)}{|\mathbf{N}(y_1)|}\right)\sin\theta_1 + \left(g_2(y_1) + y_2 \dfrac{N_2(y_1)}{|\mathbf{N}(y_1)|}\right)\cos\theta_1 \end{pmatrix},
\end{aligned}$$

it is represented in Fig. 3.7.

Deformation of the boundary of the profile : We consider a reference configuration for the structure $S_{\text{ref}} = S(0, 0)$. The boundary of this structure is described by two parametric functions :

- $\boldsymbol{\gamma}^+(\ell)$ for the extrados,
- $\boldsymbol{\gamma}^-(\ell)$ for the intrados.

The boundary of $S(\theta_1(t), \theta_2(t))$ is then described by the two parametric functions $\mathbf{X}(\theta_1, \theta_2, \boldsymbol{\gamma}^+(\ell))$ and $\mathbf{X}(\theta_1, \theta_2, \boldsymbol{\gamma}^-(\ell))$, whose expression is

$$\mathbf{X}(\theta_1, \theta_2, \boldsymbol{\gamma}^+(\ell)) = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} g_1(\gamma_1^+(\ell)) + \gamma_2^+(\ell) \dfrac{N_1(\gamma_1^+(\ell))}{|\mathbf{N}(\gamma_1^+(\ell))|} \\ g_2(\gamma_1^+(\ell)) + \gamma_2^+(\ell) \dfrac{N_2(\gamma_1^+(\ell))}{|\mathbf{N}(\gamma_1^+(\ell))|} \end{pmatrix}, \tag{3.41}$$

and the expression of $\mathbf{X}(\theta_1, \theta_2, \boldsymbol{\gamma}^-(\ell))$ is the analogy. In the sequel, for numerical tests, we consider the case of an elliptic symmetric reference domain (see Fig. 3.8).

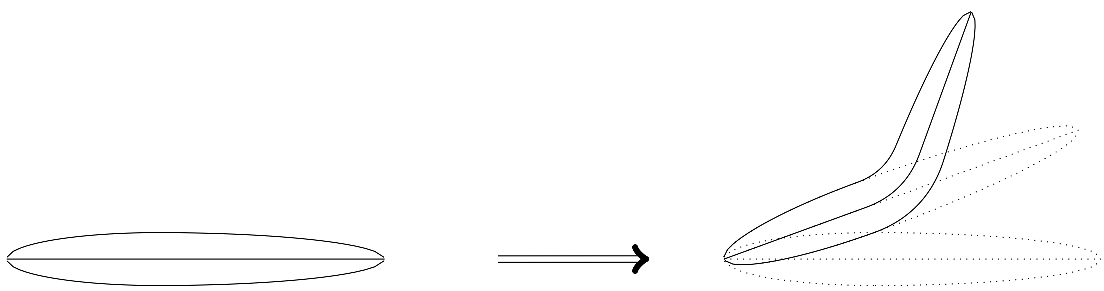


FIGURE 3.8 – The case of an ellipse.

Its boundary is given by the functions :

$$\begin{cases} \boldsymbol{\gamma}^+(\ell) = (\ell, \gamma_2^+(\ell)), \\ \boldsymbol{\gamma}^-(\ell) = (\ell, -\gamma_2^+(\ell)), \end{cases}$$

where

$$\gamma_2^+(\ell) = \begin{cases} b\sqrt{1 - \left(\dfrac{\ell - x_a}{x_a}\right)^2} & \text{if } \ell \in [0, x_a], \\ b\sqrt{1 - \left(\dfrac{\ell - x_a}{1 - x_a}\right)^2} & \text{if } \ell \in]x_a, 1]. \end{cases} \tag{3.42}$$

In the sequel, we denote $\mathbf{Y}(\theta_1, \theta_2, .)$ the inverse diffeomorphism associated to $\mathbf{X}(\theta_1, \theta_2, .)$. We do not have a priori any explicit expression of it.

Remark 3.5.1. In the considered geometry of this Chapter, the point O is at the front of the profile, while in Chapters 1 and 2, it is inside the structure.

Remark 3.5.2. The framework developed in this study can be used for more general geometries.

3.5.1.2 The diffeomorphism $\mathbf{\Phi}^{\mathbf{S}}$

We consider a stationary configuration $S_s = S(\xi_1, \xi_2)$. Let $\tilde{\Omega} \subset \Omega$ be a domain such that for every $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, we have $S(\theta_1, \theta_2) \subset \tilde{\Omega}$, see Fig. 3.9.

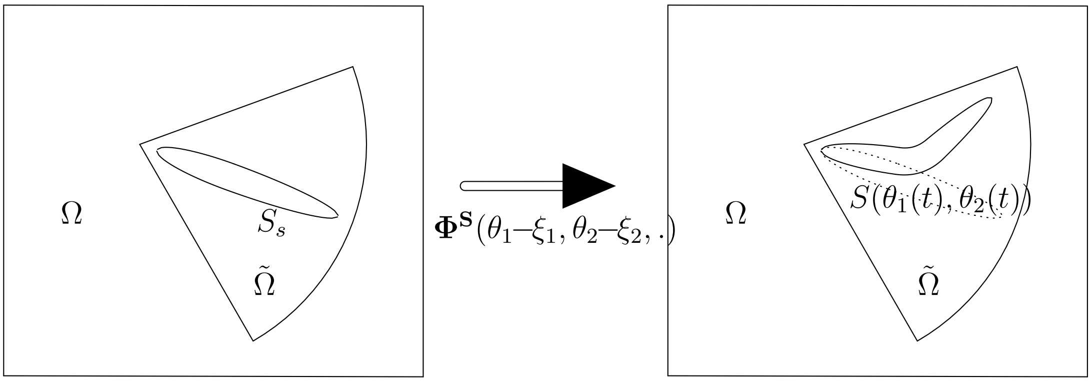


FIGURE 3.9 – The diffeomorphism $\mathbf{\Phi}^{\mathbf{S}}$.

Let $(\theta_1, \theta_2) \in \mathbb{D}_\Theta$, we consider $\mathbf{s}_{\theta_1,\theta_2}$ the solution to

$$\begin{cases} \Delta \mathbf{s}_{\theta_1,\theta_2} = 0 & \text{in } \tilde{\Omega} \backslash S_s, \\ \mathbf{s}_{\theta_1,\theta_2} = \mathbf{X}(\xi_1 + \theta_1, \xi_2 + \theta_2, \mathbf{Y}(\xi_1, \xi_2, .)) - \mathrm{Id} & \text{on } \partial S_s, \\ \mathbf{s}_{\theta_1,\theta_2} = 0 & \text{on } \partial \tilde{\Omega}. \end{cases} \tag{3.43}$$

We define the diffeomorphism $\mathbf{\Phi}^{\mathbf{S}}$ as follows

$$\forall (\theta_1, \theta_2) \in \mathbb{D}_\Theta, \ \ \forall \mathbf{y} \in \Omega, \ \ \mathbf{\Phi}^{\mathbf{S}}(\theta_1, \theta_2, \mathbf{y}) = \begin{cases} \mathbf{X}(\xi_1 + \theta_1, \xi_2 + \theta_2, \mathbf{Y}(\xi_1, \xi_2, \mathbf{y})) & \text{if } \mathbf{y} \in S_s, \\ \mathbf{y} + \mathbf{s}_{\theta_1,\theta_2}(\mathbf{y}) & \text{if } \mathbf{y} \in \tilde{\Omega} \backslash S_s, \\ \mathbf{y} & \text{if } \mathbf{y} \in \Omega \backslash \tilde{\Omega}. \end{cases} \tag{3.44}$$

Remark 3.5.3. There exists $\mathbf{X}$, θ_1 and θ_2 such that $\mathbf{\Phi}^{\mathbf{S}}(\theta_1, \theta_2, .) \notin \mathbf{W}^{3,\infty}(\Omega)$. Hence, the diffeomorphism $\mathbf{\Phi}^{\mathbf{S}}(\theta_1, \theta_2, .)$ considered in this chapter do not have the same regularity as in Chapters 1 and 2.

Remark 3.5.4. Note that this formulation is close to the usual ALE formulation. Indeed, a linearity argument shows that $\mathbf{v} = \dfrac{\mathbf{s}_{\theta_1(t+\Delta t),\theta_2(t+\Delta t)} - \mathbf{s}_{\theta_1(t),\theta_2(t)}}{\Delta t}$ is solution to

$$\begin{cases} \Delta \mathbf{v} = 0 & \text{in } \tilde{\Omega}, \\ \mathbf{v} = \dfrac{\mathbf{X}(\xi_1 + \theta_1(t+\Delta t), \xi_2 + \theta_2(t+\Delta t), \mathbf{Y}(\xi_1, \xi_2, \mathbf{y})) - \mathbf{X}(\xi_1 + \theta_1(t), \xi_2 + \theta_2(t), \mathbf{Y}(\xi_1, \xi_2, \mathbf{y}))}{\Delta t} & \text{on } \partial S_s, \\ \mathbf{v} = 0 & \text{on } \partial \tilde{\Omega}. \end{cases}$$

Hence, the diffeomorphism is obtained as the solution of a Poisson problem, like for the position of the mesh in an ALE approach.

We now show that $\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,.)$ can also be obtained by the resolution of a Poisson problem. Let $\varepsilon > 0$ small enough such that $(\theta_1+\varepsilon,\theta_2) \in \mathbb{D}_\Theta$. We have

$$\frac{\boldsymbol{\Phi}^{\mathbf{S}}(\theta_1+\varepsilon,\theta_2,\mathbf{y}) - \boldsymbol{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,\mathbf{y})}{\varepsilon}$$
$$= \begin{cases} \dfrac{\mathbf{X}(\xi_1+\theta_1+\varepsilon,\xi_2+\theta_2,\mathbf{Y}(\xi_1,\xi_2,\mathbf{y}))-\mathbf{X}(\xi_1+\theta_1,\xi_2+\theta_2,\mathbf{Y}(\xi_1,\xi_2,\mathbf{y}))}{\varepsilon} & \text{if } \mathbf{y}\in S_s, \\ \dfrac{\mathbf{s}_{\theta_1+\varepsilon,\theta_2}(\mathbf{y}) - \mathbf{s}_{\theta_1,\theta_2}(\mathbf{y})}{\varepsilon} & \text{if } \mathbf{y}\in\tilde{\Omega}\backslash S_s, \\ 0 & \text{if } \mathbf{y}\in\Omega\backslash\tilde{\Omega}, \end{cases}$$

where $\mathbf{v}_\varepsilon = \dfrac{\mathbf{s}_{\theta_1+\varepsilon,\theta_2}(\mathbf{y}) - \mathbf{s}_{\theta_1,\theta_2}(\mathbf{y})}{\varepsilon}$ is the solution to

$$\begin{cases} \Delta\mathbf{v}_\varepsilon = 0 & \text{in } \tilde{\Omega}\backslash S_s, \\ \mathbf{v}_\varepsilon = \dfrac{\mathbf{X}(\xi_1+\theta_1+\varepsilon,\xi_2+\theta_2,\mathbf{Y}(\xi_1,\xi_2,.)) - \mathbf{X}(\xi_1+\theta_1,\xi_2+\theta_2,\mathbf{Y}(\xi_1,\xi_2,.))}{\varepsilon} & \text{on } \partial S_s, \\ \mathbf{v}_\varepsilon = 0 & \text{on } \partial\tilde{\Omega}. \end{cases}$$

By passing to the limit, we get

$$\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,\mathbf{y}) = \begin{cases} \partial_{\theta_j}\mathbf{X}(\xi_1+\theta_1,\xi_2+\theta_2,\mathbf{Y}(\xi_1,\xi_2,\mathbf{y})) & \text{if } \mathbf{y}\in S_s, \\ \partial_{\theta_j}\mathbf{s}_{\theta_1,\theta_2}(\mathbf{y}) & \text{if } \mathbf{y}\in\tilde{\Omega}\backslash S_s, \\ 0 & \text{if } \mathbf{y}\in\Omega\backslash\tilde{\Omega}, \end{cases}$$

where $\partial_{\theta_j}\mathbf{s}_{\theta_1,\theta_2}$ is the solution to

$$\begin{cases} \Delta(\partial_{\theta_j}\mathbf{s}_{\theta_1,\theta_2}) = 0 & \text{in } \tilde{\Omega}\backslash S_s, \\ \partial_{\theta_j}\mathbf{s}_{\theta_1,\theta_2} = \partial_{\theta_j}\mathbf{X}(\xi_1+\theta_1,\xi_2+\theta_2,\mathbf{Y}(\xi_1,\xi_2,.)) & \text{on } \partial S_s, \\ \partial_{\theta_j}\mathbf{s}_{\theta_1,\theta_2} = 0 & \text{on } \partial\tilde{\Omega}. \end{cases}$$

Remark 3.5.5. In the sequel, we need to compute $\mathbf{Y}(\xi_1,\xi_2,.)$. In order to simplify this commputation, we set $\xi_2 = 0$. In that case, the diffeomorphism $\mathbf{Y}(\xi_1,0,.)$ is a rotation and can be computed explicitly.

The case $\xi_2 \neq 0$ could be considered, then we would not have a priori any explicit expression of $\mathbf{Y}(\xi_1,\xi_2,.)$. Its computation would be still feasible by the use of the technique described in Section 3.6.4.

3.5.2 The numerical eigenvalue problem

Let $\mathcal{T}_h$ be a conformal triangulation of the fluid domain $\mathscr{F}_s$. In the present section, we use the mesh represented in Fig. 3.1. It has 34530 cells. We use $\mathbb{P}_2$–$\mathbb{P}_1$–$\mathbb{P}_1$ Taylor–Hood spaces for $\mathbf{u}_h$, p_h and $\boldsymbol{\lambda}_h$ respectively to compute the feedback matrix $\mathbf{K}_{\delta,\omega}$,

$$\begin{aligned} V_h &= \{\mathbf{u}_h \in \mathscr{C}^0(\mathscr{F}_s) \ \text{ with } \ \mathbf{u}_{h|T} \in (\mathbb{P}_2(T))^2, \forall T \in \mathcal{T}_h\}, \\ Q_h &= \{p_h \in \mathscr{C}^0(\mathscr{F}_s) \ \text{ with } \ p_{h|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}, \\ W_h &= \{\boldsymbol{\lambda}_h \in \mathscr{C}^0(\partial S_s) + \mathscr{C}^0(\Gamma_{\mathrm{D}}) \ \text{ with } \ \boldsymbol{\lambda}_{h|\overline{T}\cap(\partial S_s\cup\Gamma_{\mathrm{D}})} \in \mathbb{P}_1(\overline{T}\cap(\partial S_s\cup\Gamma_{\mathrm{D}})), \forall T \in \mathcal{T}_h\}. \end{aligned}$$

The total number of degrees of freedom is then 153880. Note that with this discretization, the inf–sup condition (3.14) is satisfied [81].

The goal of this section is to numerically compute the feedback matrix $\mathbf{K}_{\delta,\omega}$ defined in (3.40). This requires the computation of the spectrum of the fluid–structure system.

One of the specifities of our study is that the feedback matrix is computed in the fixed domain $\mathscr{F}_s$ by studying the equations (3.13), while the time stepping process is done in the moving domain $\mathscr{F}(\theta_1(t), \theta_2(t))$ with a CutFEM approach detailed in Section 3.6 and the partitioned process described in Section 3.6.2.1.

Those two numerical processes are two different approximations of the same continuous system (3.1). That is why we expect that the feedback

$$\mathbf{h} = \mathbf{K}_{\delta,\omega}\mathbf{z}, \tag{3.45}$$

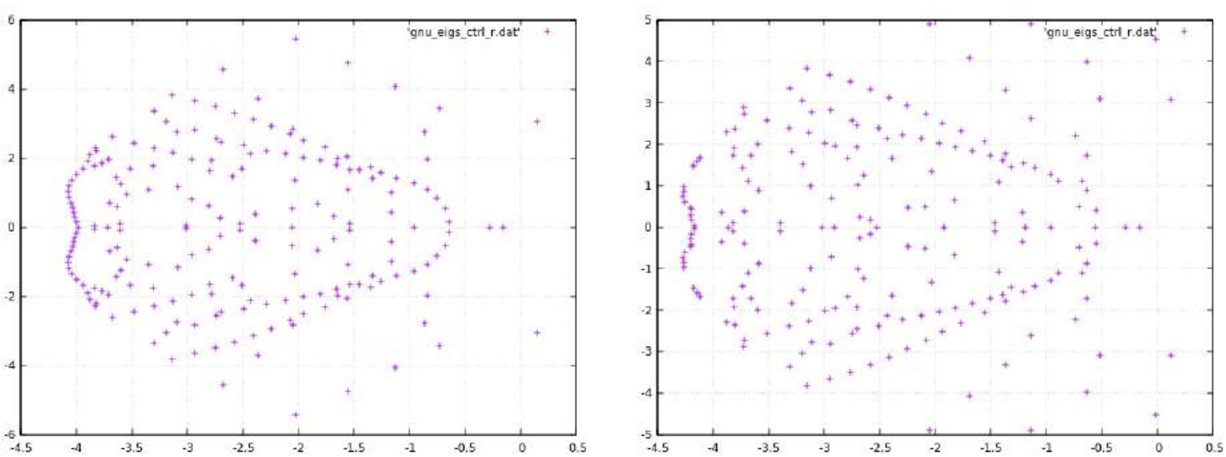
stabilizes the partitioned process for a mesh fine enough, even if it has been computed with a different approximation of the continuous problem.

Moreover, the two discretizations correspond to the same configuration given by the parameters $(\theta_1, \theta_2) = (\xi_1, \xi_2)$. In that sense, the discretized equations (3.13) are a linearization of the time–stepping process described in Section 3.6.2.1.

In the sequel we compute the spectrum of the fluid problem

$$\beta \begin{pmatrix} M_{\mathbf{uu}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \\ \mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} A_{\mathbf{uu}} & A_{\mathbf{u}p} & A_{\mathbf{u}\boldsymbol{\lambda}} \\ A_{\mathbf{u}p}^T & 0 & 0 \\ A_{\mathbf{u}\boldsymbol{\lambda}}^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \\ \mathbf{\Lambda} \end{pmatrix}, \tag{3.46}$$

where these matrices are given in Section 3.2.2.2, and the spectrum of the fluid–structure problem (3.37). The results are reported in Fig. 3.10. This is done with an Arnoldi method (size of the Arnoldi space : 400) and a shift–inverse transformation (shift : 3) implemented in the Arpack library [101].



(a) The fluid alone. (b) The fluid–structure problem in open loop.

Figure 3.10 – Spectra of the problem for $\mathcal{R}e = 120$, $k = 12$, $\rho = 5$.

The spectrum of the fluid–structure problem is very close from the one of the fluid problem. The coupling with the structure has slightly modified the eigenvalues. In both cases, there are only two conjugated unstable eigenvalues. These unstable eigenvalues will be the one considered in $\mathbb{Z}_u$ to compute the feedback matrix.

After the computation of the eigenvalues and the eigenvectors of the fluid–structure problem, we get the matrices E_u, $\widetilde{E}_u$ and Λ_u. We solve the equation (3.39), with the unstable space given by the only couple of unstable eigenvalue and with a shift $\omega = 6$.

We get the feedback matrix $\mathbf{K}_{\delta,\omega} \in \mathbb{R}^{2\times(N_\mathbf{u}+4)}$ as in (3.40). We compute the kernels of the feedback matrix as follows. We assume that the continuous feedback operator is a kernel operator, if

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \mathcal{K}_\delta \mathbf{z} = \mathcal{K}_\delta(\mathbf{u}_h\ \theta_1\ \theta_2\ \omega_1\ \omega_2).$$

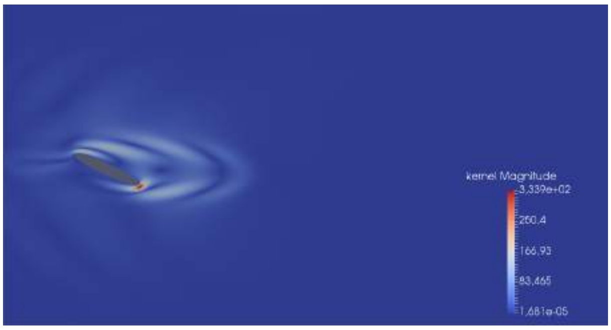
then, we have

$$h_j = \int_{\mathscr{F}_s} k_j(\mathbf{y})\mathbf{u}_h(\mathbf{y})\,\mathrm{d}\mathbf{y} + m_j(\theta_1, \theta_2, \omega_1, \omega_2),$$

where $m_j \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^2)$ and k_j is a continuous kernel. In the sequel, we want to emphasize the sensitivity of the control to the fluid state. In order to do so, we focus on the kernels k_1 and k_2. We can discretize k_j in the finite element basis V_h, we denote K_j its coordinates. We denote $\widetilde{K_1}, \widetilde{K_2} \in \mathbb{R}^{N_\mathbf{u}}$, such that $\widetilde{K}_1^T = (\mathbf{K}_{\delta,\omega})_{i=1,j=1,N_\mathbf{u}}$ and $\widetilde{K}_2^T = (\mathbf{K}_{\delta,\omega})_{i=2,j=1,N_\mathbf{u}}$. The coordinates $K_1, K_2 \in \mathbb{R}^{N_\mathbf{u}}$ of the kernels are obtained as

$$K_1 = M_{\mathbf{uu}}^{-1}\widetilde{K_1}, \qquad K_2 = M_{\mathbf{uu}}^{-1}\widetilde{K_2},$$

the discretization of these kernels are represented in Fig. 3.11.



(a) The kernel K_1.

(b) The kernel K_2.

FIGURE 3.11 – The kernels of the feedback matrix for $\mathcal{R}e = 120$, $k = 12$, $\rho = 5$.

These kernels have their highest values concentrated around the structure. Hence, the control will be the strongest when the perturbation will reach the structure.

We represent in Fig. 3.12 the spectrum of the closed loop problem

$$\beta \begin{pmatrix} M_{\mathbf{zz}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} A_{\mathbf{zz}} + B\mathbf{K}_{\delta,\omega} & A_{\mathbf{z}\boldsymbol{\eta}} \\ A_{\boldsymbol{\eta}\mathbf{z}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \boldsymbol{\eta} \end{pmatrix}.$$

We observe that the spectrum of the closed loop is the same as the one of the open loop but with all the unstable modes sent in the stable part of the spectrum.

Note that all the terms $\mathbf{L}_1$–$\mathbf{L}_6$ are heavy to implement numerically. All the computations presented in the present chapter are done with the free library GetFEM++ [134].

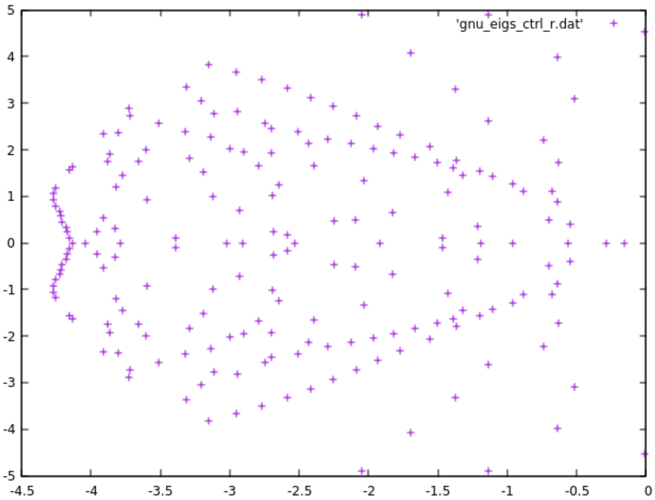


FIGURE 3.12 – The spectrum of the fluid–structure problem in closed loop.

3.6 Numerical simulations

In this section, we report the method used to run numerical simulations of the fluid–structure system in open and closed loops. Those runs are used to assess the efficiency of the feedback control procedure. Note that contrary to the previous sections, we study here the full nonlinear system. Similar studies have already been led in [2, 85] for a fluid system and in [116] for a fluid–structure interaction system.

Contrary to the latter study in which all the numerical simulations are run in the target fluid domain $\mathscr{F}_s$, i.e. in which the system (3.4) is solved numerically, we run all the numerical tests in the fluid domain at time t, i.e. we solve (3.1).

Computing the evolution of a fluid–structure interaction problem is a challenging task. The coupling between the fluid and the structure can be implemented either in a monolithic approach [84, 135], i.e. the fluid–structure system is solved as a unique problem, or in a partitioned approach [87], i.e. the fluid and the structure are solved separately and a specific method can be used for each system. In the sequel, we use a partitioned approach that seems to be more natural to take into account the control.

The main difficulty to solve (3.1) is that the computational domain, i.e. the fluid domain $\mathscr{F}(\theta_1(t), \theta_2(t))$, changes over time. We then need to apply some special techniques to handle this characteristic of the problem. A much used algorithm consists in deforming the mesh at every time step in order to make it fit the moving boundary. An efficient and classical way of deforming the mesh is the Arbitrary Lagrangian Eulerian (ALE) technique [89]. At each time step, a Poisson problem is solved in order to deform the mesh in a very smooth manner. The boundary conditions of this Poisson problem are imposed by the deformation of the structure. A non exhaustive list of works using this technique can be found in [100, 137, 139]. A review of partitioned approachs that use a conforming mesh can be found in [87].

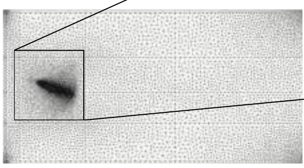
The conforming methods have some drawbacks, for instance an actual remeshing cannot be avoided when the deformations are too large and it is then time–consuming. That is why, in the present study, we are interested in a non–conforming method. We use a fictitious domain approach, i.e. the simulation can be run with the boundary that cuts the mesh. Several methods are included in this framework, for instance the immersed boundary method [126, 55, 113] or the penalization method [99, 110, 5].

In the present study, we use a XFEM type method that can be found in the litterature under the name of cutFEM. This kind of methods has been developed in the context of crack propagation in fatigue mechanics [82, 56]. The main characteristic of this method is the use of a level–set function to locate an interface in the domain and an enrichment of the finite element basis with functions depending on the position of the interface. In the crack propagation context, the interface is the crack, while in the fluid–structure context the interface is the boundary between the fluid and the structure.

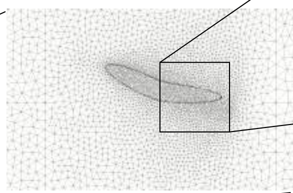
In the present study, we use 'cut' elements, i.e. some elements of the finite element method are cut by the boundary of the fluid domain. We then consider only their surface that is contained in the fluid domain. It has already been used for instance in [21, 44, 41, 40, 92] and in the context of fluid–structure interaction problems in [3, 109, 142]. Recent progress about this method can be found in [27]. A review of the cutFEM technique can be found in [93].

3.6.1 The basis functions

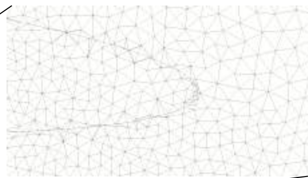
We define a background mesh covering $\Omega = \mathscr{F}(\theta_1, \theta_2) \cup S(\theta_1, \theta_2)$. The interface between the fluid and the solid can arbitrary cut this mesh, see for instance Fig. 3.13.



(a) Global domain.



(b) Near the structure.



(c) Near the interface for integration only.

FIGURE 3.13 – The fictitious domain.

Some cells of this mesh are in the fluid only, some others are in the structure only and the other cells are shared between the fluid and the structure. The mesh is fixed and the structure moves over time.

We use Lagrange multipliers to enforce the Dirichlet boundary conditions. We define a triangulation $\mathcal{T}_h$ of Ω and a background finite element method with $\mathbb{P}_2$–$\mathbb{P}_1$–$\mathbb{P}_1$ Taylor–Hood elements for the velocity, the pressure and the multipliers respectively,

$$\widetilde{V_h} = \{\mathbf{u}_h \in \mathscr{C}^0(\Omega) \text{ with } \mathbf{u}_{h|T} \in (\mathbb{P}_2(T))^2, \quad \forall T \in \mathcal{T}_h\},$$

$$\widetilde{Q_h} = \{p_h \in \mathscr{C}^0(\Omega) \text{ with } p_{h|T} \in \mathbb{P}_1(T), \quad \forall T \in \mathcal{T}_h\},$$

$$\widetilde{W_h} = \{\boldsymbol{\lambda}_h \in \mathscr{C}^0(\Omega) \text{ with } \boldsymbol{\lambda}_{h|T} \in (\mathbb{P}_1(T))^2, \quad \forall T \in \mathcal{T}_h\}.$$

The basis functions that are considered in the sequel are traces of the background basis functions of $\widetilde{V_h}$, $\widetilde{Q_h}$ and $\widetilde{W_h}$. These traces are taken over the fluid domain $\mathscr{F}(\theta_1,\theta_2)$ for the basis functions of the velocity and the pressure and on the interface $\partial S(\theta_1,\theta_2)$ for the Lagrange multipliers. More precisely, we consider

$$V_h^n = \{\mathbf{u}_{h|\mathscr{F}(\theta_1^n,\theta_2^n)} \text{ with } \mathbf{u}_h \in \widetilde{V_h}\},$$

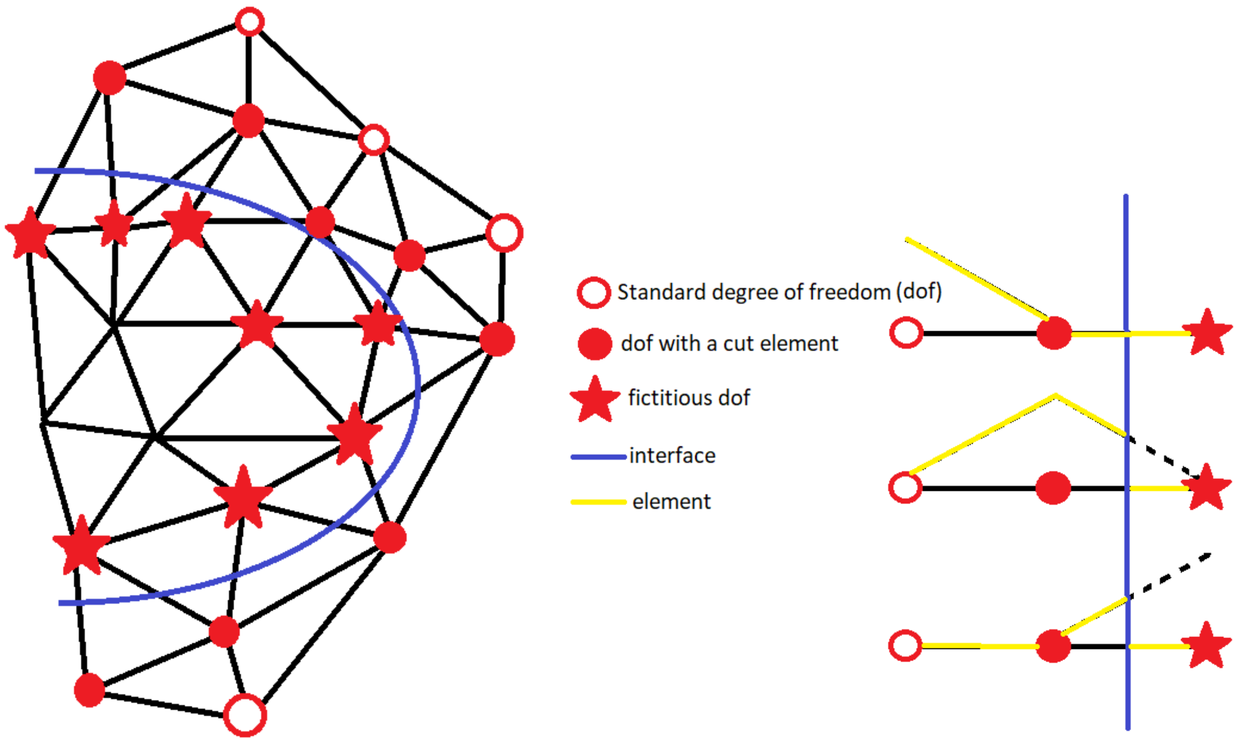
$$Q_h^n = \{p_{h|\mathscr{F}(\theta_1^n,\theta_2^n)} \text{ with } p_h \in \widetilde{Q_h}\},$$

$$W_h^n = \{\boldsymbol{\lambda}_{h|\partial S(\theta_1^n,\theta_2^n)} \text{ with } \boldsymbol{\lambda}_h \in \widetilde{W_h}\},$$

where the superscript n emphasizes the dependence of these spaces on the parameters $(\theta_1^n,\theta_2^n) = (\theta_1(t^n),\theta_2(t^n))$ at time t^n.

A basis for the space V_h^n (resp. Q_h^n) is obtained as follows. The elements of $\widetilde{V_h}$ (resp. $\widetilde{Q_h}$) whose trace on $\mathscr{F}(\theta_1^n,\theta_2^n)$ is null are discarded. The elements that are cut by the interface become 'cut elements', i.e. we consider only their trace over $\mathscr{F}(\theta_1^n,\theta_2^n)$. The elements strictly inside $\mathscr{F}(\theta_1^n,\theta_2^n)$ are left unchanged. In that way, we have built a basis of V_h^n (resp. Q_h^n). We represent these elements in Fig. 3.14b for $\mathbb{P}1$ elements in a 1D configuration.

We represent the degrees of freedom associated to a 2D–$\mathbb{P}1$ basis in Fig. 3.14a.



(a) Degrees of freedom represented on a mesh.

(b) 1D representation of the cut elements.

FIGURE 3.14 – The degrees of freedom for $\mathbb{P}1$ elements.

The degrees of freedom that are away from the structure are associated to unchanged elements. The degrees of freedom inside the structure that are not attached to any cut element are discarded. The degrees of freedom inside the structure that are attached to cut elements are called 'fictitious degrees of freedom'. They require specific attention in the sequel since their values do not correspond to any actual fluid velocity or pressure values.

Hence, the classical finite element method basis for the fluid domain has been completed by cut elements near $\partial S(\theta_1^n, \theta_2^n)$. In that sense, we can consider this method as a XFEM method.

It is a little bit more delicate to find a basis for the space W_h^n. Indeed, the traces on $\partial S(\theta_1^n, \theta_2^n)$ of the basis functions of $\widetilde{W_h}$ may be linearly dependent, especially if $\partial S(\theta_1^n, \theta_2^n)$ is locally rectilinear. In practice, the redundant functions are eliminated in the GetFEM++ code.

Remark 3.6.1. These Taylor–Hood elements coincide with the elements of Section 3.5.2 in case of a conformal mesh.

3.6.2 The time stepping process

The weak formulation of problem (3.1) that we use in the sequel is the following one.

Find $(\theta_1, \theta_2, \omega_1, \omega_2) \in \mathrm{H}^2(0,T;\mathbb{D}_\Theta) \times \mathrm{H}^1(0,T;\mathbb{R}^2)$
and $(\mathbf{u}, p, \boldsymbol{\lambda}) \in (\mathrm{H}^1(0,T;\mathbf{L}^2(\mathscr{F}(\theta_1,\theta_2))) \cap \mathrm{L}^2(0,T;\mathbf{H}^1(\mathscr{F}(\theta_1,\theta_2))))$
$\times \mathrm{L}^2(0,T;\mathrm{L}^2(\mathscr{F}(\theta_1,\theta_2))) \times \mathrm{L}^2\Big(0,T;\Big(\mathbf{H}^{-1/2}(\partial S(\theta_1,\theta_2)) \times \mathbf{H}^{-1/2}(\Gamma_\mathrm{D})\Big)\Big)$ such that

$$\begin{cases} \displaystyle\int_{\mathscr{F}(\theta_1,\theta_2)} \frac{\partial \mathbf{u}}{\partial t}\cdot\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} + \frac{\nu}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T) : (\nabla\mathbf{v}+\nabla\mathbf{v}^T) - p\mathrm{div}\,\mathbf{v}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathrm{D}\cup\partial S(\theta_1,\theta_2)} \boldsymbol{\lambda}\cdot\mathbf{v}\,\mathrm{d}\gamma_x = 0, \\ \displaystyle\int_{\mathscr{F}(\theta_1,\theta_2)} q\mathrm{div}\,\mathbf{u}\,\mathrm{d}\mathbf{x} = 0, \\ \displaystyle\int_{\Gamma_\mathrm{D}\cup\partial S(\theta_1,\theta_2)} \mathbf{u}\cdot\boldsymbol{\mu}\,\mathrm{d}\gamma_x = \int_{\Gamma_\mathrm{i}} (\mathbf{u}^i + \mathbf{u}^p)\cdot\boldsymbol{\mu}\,\mathrm{d}\gamma_x + \int_{\partial S(\theta_1,\theta_2)} \sum_j \omega_j \partial_{\theta_j}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\cdot\boldsymbol{\mu}\,\mathrm{d}\gamma_x, \end{cases} \tag{3.47}$$

for every $(\mathbf{v}, q, \boldsymbol{\mu}) \in \mathbf{H}^1(\mathscr{F}(\theta_1,\theta_2)) \times \mathrm{L}^2(\mathscr{F}(\theta_1,\theta_2)) \times \Big(\mathbf{H}^{-1/2}(\partial S(\theta_1,\theta_2)) \times \mathbf{H}^{-1/2}(\Gamma_\mathrm{D})\Big)$ and

$$\begin{cases} \mathcal{M}_{\theta_1,\theta_2}\begin{pmatrix}\dot\omega_1\\ \dot\omega_2\end{pmatrix} = \left(\displaystyle\int_{\partial S(\theta_1,\theta_2)} \boldsymbol{\lambda}(\gamma_x)\cdot\partial_{\theta_j}\mathbf{X}(\theta_1,\theta_2,\mathbf{Y}(\theta_1,\theta_2,\gamma_x))\,\mathrm{d}\gamma_x\right)_{j=1,2} + \mathbf{M_I}(\theta_1,\theta_2,\omega_1,\omega_2) + \mathbf{h}, \\ \dot\theta_1 = \omega_1, \\ \dot\theta_2 = \omega_2. \end{cases}$$

It is completed with initial conditions.

In the sequel, we add a stabilization term in the fluid part of this variational formulation and we use a partitioned approach (see [62, 42, 43]), which means that we treat the update with two steps : a fluid step and a structure step.

3.6.2.1 The partitioned process

At each time step, we use the procedure described in Algorithm 1.

Note that the coupling between the fluid and the structure is explicit. This is less time–consuming than an implicit coupling and is still very efficient [63, 97].

Algorithm 1 The splitting scheme used

Require: $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\lambda}_h^n, \theta_1^{n-1}, \theta_2^{n-1}, \theta_1^n, \theta_2^n, \omega_1^n, \omega_2^n)$
1 Compute $\mathbf{h}$ (if a closed loop run is considered), see Section 3.6.2.6.
2 Compute $(\theta_1^{n+1}, \theta_2^{n+1}, \omega_1^{n+1}, \omega_2^{n+1})$ with the structure step (3.48).
3 Update $\mathscr{F}(\theta_1^{n+1}, \theta_2^{n+1})$, V_h^{n+1}, Q_h^{n+1} and W_h^{n+1}.
4 Compute the matrices M^{n+1}, A^{n+1} and F^{n+1}.
5 Compute $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\lambda}_h^{n+1})$ with the fluid step (3.51).
6 Compute the next time step Δt with (3.53).

3.6.2.2 The step for the structure

We use a backward difference scheme to approximate the velocity and displacement of the structure. We assume that θ_1^{n-1}, θ_2^{n-1}, θ_1^n, θ_2^n, ω_1^n, ω_2^n and $\boldsymbol{\lambda}_h^n$ are known and we compute

$$\begin{cases} \begin{pmatrix}\theta_1^{n+1}\\ \theta_2^{n+1}\end{pmatrix} = 2\begin{pmatrix}\theta_1^{n}\\ \theta_2^{n}\end{pmatrix} - \begin{pmatrix}\theta_1^{n-1}\\ \theta_2^{n-1}\end{pmatrix} \\ \qquad\qquad +(\Delta t)^2 \mathcal{M}^{-1}_{\theta_1^n,\theta_2^n}\left(\mathbf{M_A}(\theta_1^n,\theta_2^n,\boldsymbol{\lambda}_h^n) + \mathbf{M_I}(\theta_1^n,\theta_2^n,\omega_1^n,\omega_2^n) + \mathbf{h}^n - k\begin{pmatrix}\theta_1^n-\xi_1\\ \theta_2^n-\xi_2\end{pmatrix}\right), \\ \begin{pmatrix}\omega_1^{n+1}\\ \omega_2^{n+1}\end{pmatrix} = \begin{pmatrix}\omega_1^{n}\\ \omega_2^{n}\end{pmatrix} + \Delta t \mathcal{M}^{-1}_{\theta_1^n,\theta_2^n}\left(\mathbf{M_A}(\theta_1^n,\theta_2^n,\boldsymbol{\lambda}_h^n) + \mathbf{M_I}(\theta_1^n,\theta_2^n,\omega_1^n,\omega_2^n) + \mathbf{h}^n - k\begin{pmatrix}\theta_1^n-\xi_1\\ \theta_2^n-\xi_2\end{pmatrix}\right). \end{cases} \tag{3.48}$$

3.6.2.3 The stabilization term

In order to guarantee the optimal convergence of the Lagrange multipliers, we add to the approximated variational problem a stabilization term. Such a technique has been introduced in [14] for conformal meshes. It has been adapted for non conformal meshes in [82] for the Poisson problem and in [52] for the stationary Stokes problem.

This term can be chosen in several ways [65], for instance we can add in (3.47) the term

$$-\gamma_0 h \int_{\partial S(\theta_1,\theta_2)} (\boldsymbol{\lambda}_h + \sigma_F(\mathbf{u}_h, p_h)\mathbf{n}_{\theta_1,\theta_2}) \cdot (\boldsymbol{\mu}_h + \sigma_F(\mathbf{v}_h, q_h)\mathbf{n}_{\theta_1,\theta_2})\, \mathrm{d}\gamma_x, \tag{3.49}$$

with a mesh–independent constant $\gamma_0 > 0$.

The derivation of this term uses an augmented Lagrangian approach [52, Section 2].

The goal of this term is to enhance the convergence of $\boldsymbol{\lambda}_h$ to $-\sigma_F(\mathbf{u}_h, p_h)\mathbf{n}_{\theta_1,\theta_2}$. Note that a good approximation of $\boldsymbol{\lambda}_h$ is very important since this term is used to compute the forces of the fluid acting on the structure.

The choice of the parameter γ_0 has to respond to a compromise between the coercivity of the system and the weight of the stabilization term [52].

This stabilization term enables us to recover the optimal convergence rate for the multiplier $\boldsymbol{\lambda}_h$, provided that all the cut mesh elements T are such that $\mathscr{F}(\theta_1,\theta_2)\cap T$ is a big enough portion of T. We call 'bad elements' the elements T such that $\mathscr{F}(\theta_1,\theta_2)\cap T$ is too small compared with T. More precisely, we have the following definition and assumption.

Assumption A. We fix a threshold $\alpha_{\min} \in (0,1]$ and declare any cut element T a good element (resp. a bad element) if $\frac{|T\cap\mathscr{F}(\theta_1,\theta_2)|}{|T|} \geq \alpha_{\min}$ (resp. $\frac{|T\cap\mathscr{F}(\theta_1,\theta_2)|}{|T|} < \alpha_{\min}$). We assume that one can choose for any bad element T a 'good neighbor' $T' \in \mathcal{T}_h$ such that $\frac{|T'\cap\mathscr{F}(\theta_1,\theta_2)|}{|T'|} \geq \alpha_{\min}$ and such that T and T' share at least one node (see Fig. 3.15).

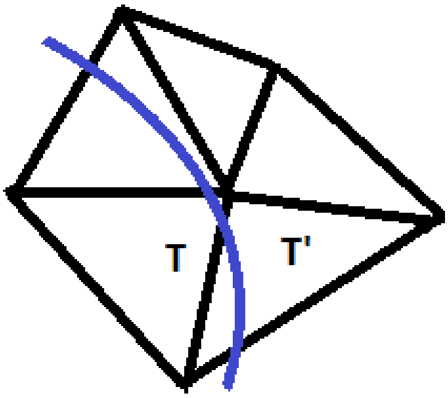


FIGURE 3.15 – A bad element T and a good neighbor T'.

We now define the 'robust reconstruction' of a function.

Definition 3.6.2. For any $\mathbf{v}_h \in V_h^{n+1}$, we define $\widehat{\mathbf{v}_h}$, the robust reconstruction of $\mathbf{v}_h$, on any element T of $\mathcal{T}_h$ such that $T \cap \mathscr{F}(\theta_1, \theta_2) \neq \emptyset$ by
- $\widehat{\mathbf{v}_h} = \mathbf{v}_h$ on T if T is a good element,
- $(\widehat{\mathbf{v}_h})_{|T} = (\mathbf{v}_h)_{|T'}$ on T if T is a bad element. Here, T' is a good neighbor of T whose existence is given by Assumption A. The relation should be understood in the sense that $(\widehat{\mathbf{v}_h})_{|T}$ is taken as the same polynomial as $(\mathbf{v}_h)_{|T'}$.

Remark 3.6.3. We define $\widehat{p_h}$ in the same manner. See [65] for more information.

In order to ensure that only good elements are considered, we use the robust reconstruction in the stabilization term that becomes

$$-\gamma_0 h \int_{\partial S(\theta_1,\theta_2)} (\boldsymbol{\lambda}_h + \sigma_F(\widehat{\mathbf{u}_h}, \widehat{p_h})\mathbf{n}_{\theta_1,\theta_2}) \cdot (\boldsymbol{\mu}_h + \sigma_F(\widehat{\mathbf{v}_h}, \widehat{q_h})\mathbf{n}_{\theta_1,\theta_2}) \, \mathrm{d}\gamma_x.$$

More information about this stabilization term can be found in [65, 52].

3.6.2.4 The step for the fluid

We assume that $\mathbf{u}_h^n$, θ_1^{n+1}, θ_2^{n+1}, ω_1^{n+1} and ω_2^{n+1} are known at this point. We use an Euler scheme to estimate the time derivative of the fluid velocity. The nonlinear term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is approximated with the semi–implicit term $(\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^{n+1}$. Note that in order to ensure the stability of the scheme, we use the same CFL condition as for the explicit discretization $(\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^n$ [127, 83] (see Section 3.6.2.7). Moreover, because of the adherence between the fluid and the structure, this CFL condition also constraints the displacement of the structure which can not cross several cells during one time step. The fully discretized variational formulation for the fluid is the following one.

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\lambda}_h^{n+1}) \in V_h^{n+1} \times Q_h^{n+1} \times W_h^{n+1}$ such that

$$\begin{cases} \displaystyle\int_{\mathscr{F}(\theta_1^{n+1},\theta_2^{n+1})} \frac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t} \cdot \mathbf{v}_h + (\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^{n+1} \cdot \mathbf{v}_h \\ \qquad\qquad + \dfrac{\nu}{2}(\nabla \mathbf{u}_h^{n+1} + (\nabla \mathbf{u}_h^{n+1})^T) : (\nabla \mathbf{v}_h + \nabla \mathbf{v}_h^T) - p_h^{n+1} \mathrm{div}\, \mathbf{v}_h \, \mathrm{d}\mathbf{x} \\ \quad + \displaystyle\int_{\Gamma_{\mathrm{D}} \cup \partial S(\theta_1^{n+1},\theta_2^{n+1})} \boldsymbol{\lambda}_h^{n+1} \cdot \mathbf{v}_h \, \mathrm{d}\gamma_x - \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \left(\boldsymbol{\lambda}_h^{n+1} + \sigma_F(\widehat{\mathbf{u}_h^{n+1}}, \widehat{p_h^{n+1}}) \mathbf{n}_{\theta_1,\theta_2} \right) \\ \qquad\qquad \cdot \nu (\nabla \widehat{\mathbf{v}_h} + \nabla \widehat{\mathbf{v}_h}^T) \mathbf{n}_{\theta_1,\theta_2} \, \mathrm{d}\gamma_x = 0, \\ \displaystyle\int_{\mathscr{F}(\theta_1^{n+1},\theta_2^{n+1})} q_h \mathrm{div}\, \mathbf{u}_h^{n+1} \, \mathrm{d}\mathbf{x} + \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \left(\boldsymbol{\lambda}_h + \sigma_F(\widehat{\mathbf{u}_h^{n+1}}, \widehat{p_h^{n+1}}) \mathbf{n}_{\theta_1,\theta_2} \right) \widehat{q_h} \mathbf{n}_{\theta_1,\theta_2} \, \mathrm{d}\gamma_x = 0, \\ \displaystyle\int_{\Gamma_{\mathrm{D}} \cup \partial S(\theta_1^{n+1},\theta_2^{n+1})} \mathbf{u}_h^{n+1} \cdot \boldsymbol{\mu}_h \, \mathrm{d}\gamma_x - \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \left(\boldsymbol{\lambda}_h + \sigma_F(\widehat{\mathbf{u}_h^{n+1}}, \widehat{p_h^{n+1}}) \mathbf{n}_{\theta_1,\theta_2} \right) \cdot \boldsymbol{\mu}_h \, \mathrm{d}\gamma_x \\ \quad = \displaystyle\int_{\Gamma_{\mathrm{i}}} (\mathbf{u}^i + \mathbf{u}^p) \cdot \boldsymbol{\mu}_h \, \mathrm{d}\gamma_x + \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \sum_j \omega_j \partial_{\theta_j} \mathbf{X}(\theta_1^{n+1}, \theta_2^{n+1}, \mathbf{Y}(\theta_1^{n+1}, \theta_2^{n+1}, \gamma_x)) \cdot \boldsymbol{\mu}_h \, \mathrm{d}\gamma_x, \end{cases} \tag{3.50}$$

for every $(\mathbf{v}_h, q_h, \boldsymbol{\mu}_h) \in V_h^{n+1} \times Q_h^{n+1} \times W_h^{n+1}$.

We denote respectively $(\mathcal{U}_k)$, $(\mathcal{P}_k)$, $(\mathcal{W}_k)$ the basis functions of V_h^{n+1}, Q_h^{n+1}, W_h^{n+1} and $\mathbf{U}$, $\mathbf{P}$, $\boldsymbol{\Lambda}$ the coefficients of $\mathbf{u}_h$, p_h, $\boldsymbol{\lambda}_h$ in those basis. So that we assume that $\mathbf{U}^n$, θ_1^{n+1}, θ_2^{n+1}, ω_1^{n+1} and ω_2^{n+1} are known and the above variational formulation reads :

$$(M^{n+1} + \Delta t A^{n+1}) \mathbf{Z}^{n+1} = \Delta t \mathbf{F}^{n+1} + M^{n+1} \mathbf{Z}^n, \tag{3.51}$$

where those vectors and matrices are given by

$$M^{n+1} = \begin{pmatrix} M_{\mathbf{uu}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{n+1} = \begin{pmatrix} A_{\mathbf{uu}} & A_{\mathbf{u}p} & A_{\mathbf{u}\boldsymbol{\lambda}} \\ A_{\mathbf{u}p}^T & A_{pp} & A_{p\boldsymbol{\lambda}} \\ A_{\mathbf{u}\boldsymbol{\lambda}}^T & A_{p\boldsymbol{\lambda}}^T & A_{\boldsymbol{\lambda\lambda}} \end{pmatrix}, \quad \mathbf{Z}^k = \begin{pmatrix} \mathbf{U}^k \\ \mathbf{P}^k \\ \boldsymbol{\Lambda}^k \end{pmatrix} \quad \text{and} \quad \mathbf{F}^{n+1} = \begin{pmatrix} 0 \\ 0 \\ F_{\boldsymbol{\lambda}} \end{pmatrix},$$

with

$$\begin{aligned} (M_{\mathbf{uu}})_{jk} &= \int_{\mathscr{F}(\theta_1^{n+1},\theta_2^{n+1})} \mathcal{U}_j \cdot \mathcal{U}_k \, \mathrm{d}\mathbf{x}, \\ (A_{\mathbf{uu}})_{jk} &= \int_{\mathscr{F}(\theta_1^{n+1},\theta_2^{n+1})} (\mathbf{u}_h^n \cdot \nabla)\mathcal{U}_k \cdot \mathcal{U}_j + \frac{\nu}{2}(\nabla \mathcal{U}_j + \nabla \mathcal{U}_j^T) : (\nabla \mathcal{U}_k + \nabla \mathcal{U}_k^T) \, \mathrm{d}\mathbf{x} \\ &\qquad - \nu^2 \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} (\nabla \widehat{\mathcal{U}_j} + \nabla \widehat{\mathcal{U}_j}^T) \mathbf{n}_{\theta_1,\theta_2} \cdot (\nabla \widehat{\mathcal{U}_k} + \nabla \widehat{\mathcal{U}_k}^T) \mathbf{n}_{\theta_1,\theta_2} \, \mathrm{d}\gamma_x, \\ (A_{\mathbf{u}p})_{jk} &= -\int_{\mathscr{F}(\theta_1^{n+1},\theta_2^{n+1})} \mathcal{P}_k \mathrm{div}\, \mathcal{U}_j \, \mathrm{d}\mathbf{x} + \nu \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \widehat{\mathcal{P}_k} \mathbf{n}_{\theta_1,\theta_2} \cdot (\nabla \widehat{\mathcal{U}_j} + \nabla \widehat{\mathcal{U}_j}^T) \mathbf{n}_{\theta_1,\theta_2} \, \mathrm{d}\gamma_x, \\ (A_{\mathbf{u}\boldsymbol{\lambda}})_{jk} &= -\int_{\Gamma_{\mathrm{D}} \cup \partial S(\theta_1^{n+1},\theta_2^{n+1})} \mathcal{U}_j \cdot \mathcal{W}_k \, \mathrm{d}\gamma_x - \nu \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \mathcal{W}_k \cdot (\nabla \widehat{\mathcal{U}_j} + \nabla \widehat{\mathcal{U}_j}^T) \mathbf{n}_{\theta_1,\theta_2} \, \mathrm{d}\gamma_x, \\ (A_{pp})_{jk} &= -\gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \widehat{\mathcal{P}_j} \widehat{\mathcal{P}_k} \, \mathrm{d}\gamma_x, \\ (A_{p\boldsymbol{\lambda}})_{jk} &= \gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \widehat{\mathcal{P}_j} \mathbf{n}_{\theta_1,\theta_2} \cdot \mathcal{W}_k \, \mathrm{d}\gamma_x, \\ (A_{\boldsymbol{\lambda\lambda}})_{jk} &= -\gamma_0 h \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \mathcal{W}_j \cdot \mathcal{W}_k \, \mathrm{d}\gamma_x, \end{aligned}$$

and

$$(F_{\boldsymbol{\lambda}})_k = \int_{\Gamma_{\mathrm{i}}} (\mathbf{u}^i + \mathbf{u}^p) \cdot \mathcal{W}_k \, \mathrm{d}\gamma_x + \int_{\partial S(\theta_1^{n+1},\theta_2^{n+1})} \sum_j \omega_j^{n+1} \partial_{\theta_j} \mathbf{X}(\theta_1^{n+1}, \theta_2^{n+1}, \mathbf{Y}(\theta_1^{n+1}, \theta_2^{n+1}, \gamma_x)) \cdot \mathcal{W}_k \, \mathrm{d}\gamma_x.$$

Remark 3.6.4. The diffeomorphism $\mathbf{Y}$ appears in the expression of $F_{\boldsymbol{\lambda}}$. The computation of this term uses the technique explained in Section 3.6.4 afterwards.

Remark 3.6.5. When the mesh is conformal and $\gamma_0 = 0$, we have a classical Taylor–Hood FEM (see Section 3.5.2) and we recover the classical matrix $A = \begin{pmatrix} A_{\mathbf{uu}} & A_{\mathbf{u}p} & A_{\mathbf{u}\boldsymbol{\lambda}} \\ A^T_{\mathbf{u}p} & 0 & 0 \\ A^T_{\mathbf{u}\boldsymbol{\lambda}} & 0 & 0 \end{pmatrix}$.

3.6.2.5 Computation of the previous time values for the fluid velocity

When solving (3.51) we need $\mathbf{U}^n$, the coordinates of $\mathbf{u}_h^n$ written in the basis of the space V_h^{n+1}. Far from the interface, where there are no cut elements, those coordinates are the same as the coordinates of $\mathbf{u}_h^n$ written in V_h^n. However, near the interface, the shape of the cut elements can change at each time step. Then, the coordinates can also change.

Moreover, some new degrees of freedom can appear since some nodes that were attached only to discarded elements at $t = t^n$ can be attached to a cut element at $t = t^{n+1}$, see Fig. 3.16. We have to give a numerical value to such nodes. The method chosen is to give them the velocity of the structure.

We represent in Fig. 3.16 the apparition of new degrees of freedom in the fluid in the time marching process.

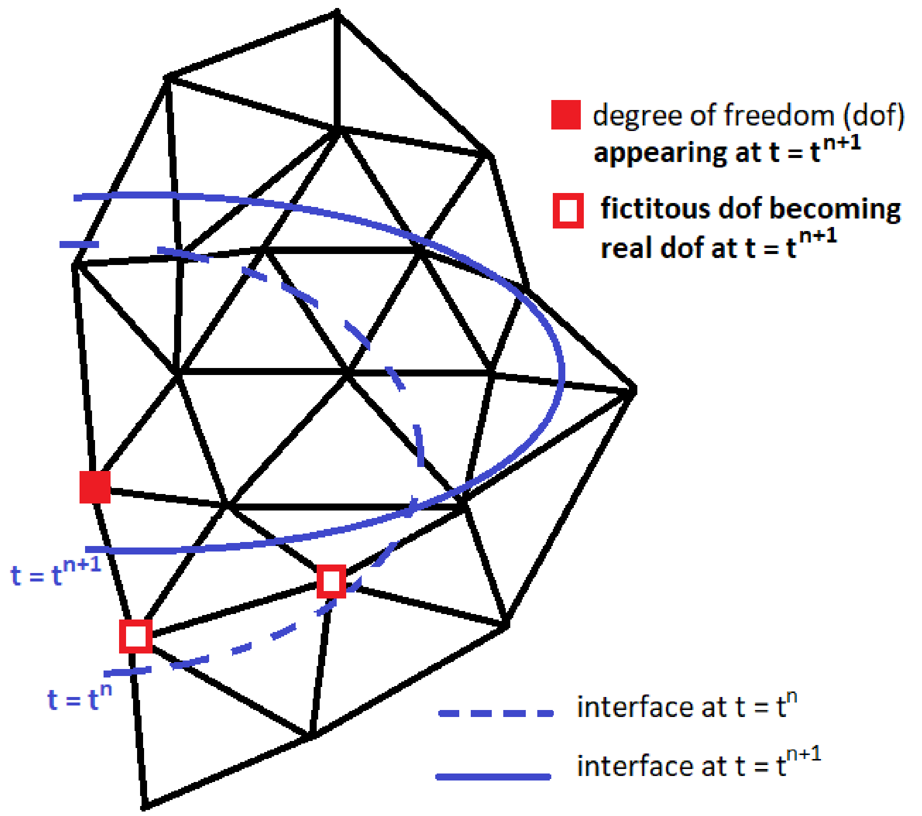


FIGURE 3.16 – The update of the freshly cleared cells.

The value of the velocity $\mathbf{u}_h^n$ at a node for the time t^n is computed in the following way. If the node was in the fluid for the time t^n, we assign the value of the fluid, if the node was in the structure, we assign the velocity of the structure.

The velocity of the structure

$$\mathbf{v}_s^n(\mathbf{x}) = \omega_1^n \partial_{\theta_1} \mathbf{X}(\theta_1^n, \theta_2^n, \mathbf{Y}(\theta_1^n, \theta_2^n, \mathbf{x})) + \omega_2^n \partial_{\theta_2} \mathbf{X}(\theta_1^n, \theta_2^n, \mathbf{Y}(\theta_1^n, \theta_2^n, \mathbf{x})),$$

requires a special treatment to be computed because of the fact it uses the inverse diffeomorphism $\mathbf{Y}$ (see Section 3.6.4).

3.6.2.6 Computation of the control function

The main originality of our work compared with other fluid–structure interaction problem stabilization studies is that the simulation is run in the actual domain $\mathscr{F}(\theta_1(t), \theta_2(t))$ instead of the reference domain $\mathscr{F}_s$. However, the feedback matrix $\mathbf{K}_{\delta,\omega}$ (see (3.40)) has been computed in $\mathscr{F}_s$ (see Section 3.5), then the feedback control is computed as $\mathbf{h}^n = \mathbf{K}_{\delta,\omega}\mathbf{z}^n$, where $\mathbf{z}^n = (\widetilde{\mathbf{U}}^n \ \theta_1^n \ \theta_2^n \ \omega_1^n \ \omega_2^n)^T$, and $\widetilde{\mathbf{U}}^n$ are the coordinates in V_h of

$$\mathbf{v}_h = \mathrm{cof}(\nabla \mathbf{\Phi}^{\mathbf{S}}(\theta_1 - \xi_1, \theta_2 - \xi_2, .))^T \mathbf{u}_h \circ \mathbf{\Phi}^{\mathbf{S}}(\theta_1 - \xi_1, \theta_2 - \xi_2, .). \tag{3.52}$$

In order to get the value of $\mathbf{v}_h$, for every base node $\mathbf{y}$ of the mesh in $\mathscr{F}_s$, we compute $\mathbf{x} = \mathbf{\Phi}^{\mathbf{S}}(\theta_1, \theta_2, \mathbf{y})$. The point $\mathbf{x}$ belongs to $\mathscr{F}(\theta_1(t), \theta_2(t))$, we then obtain the value of $\mathbf{u}_h(\mathbf{x})$ by interpolation. We assign this value to $\mathbf{y}$ and we get an approximation of $\mathbf{u}_h \circ \mathbf{\Phi}^{\mathbf{S}}(\theta_1, \theta_2, .)$ in $\mathscr{F}_s$. This enables us to compute $\mathbf{h}$ at each time step. This process is represented in Fig. 3.17.

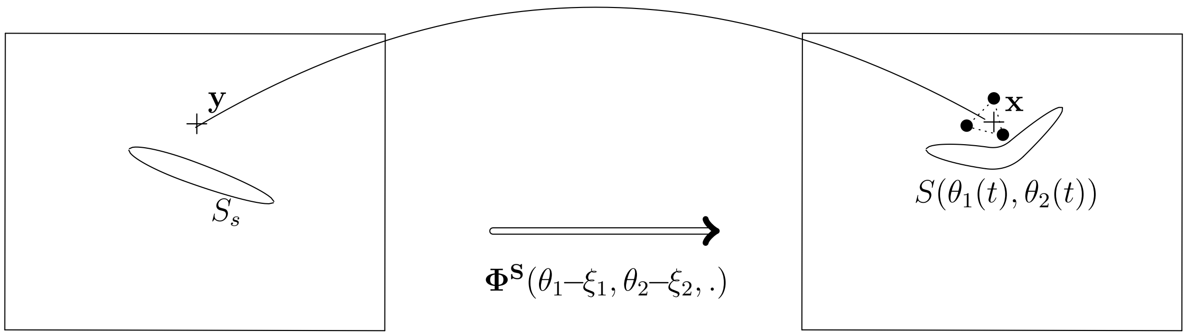


FIGURE 3.17 – Interpolation to compute the velocity in the reference domain.

3.6.2.7 A CFL condition

The time step Δt is computed as the minimum between a CFL condition and a maximum time step, i.e.

$$\Delta t^{n+1} = \min \left(cfl \times \frac{h}{V_{\max}^n}, \Delta t_{\max} \right), \tag{3.53}$$

where $V_{\max}^n$ is the maximum velocity of the fluid in $\mathscr{F}(\theta_1^n, \theta_2^n)$, $cfl \in (0,1)$ is a CFL coefficient and $\Delta t_{\max}$ is a fixed upper bound for the time step. Note that this CFL condition meets two constraints : the fluid must not cross several cells in one time step and the structure must not cross several cells in one time step (otherwise the explicit coupling during the fluid and the structure could encounter difficulties).

3.6.3 Level–set function and integration method over the cut elements

The matrices of Section 3.6.2.4 are computed via some integrations over $\mathscr{F}(\theta_1, \theta_2)$ and $\partial S(\theta_1, \theta_2)$. The associated integration methods need a well–defined interface $\partial S(\theta_1, \theta_2)$ and a method to integrate functions over the cut elements. The interface is defined as the null level of a level–set function and the integration over the cut cells is done by dividing those cells into sub–cells by the use of the 'qhull' library [13].

3.6.3.1 Computation of the level–set function

Defining the interface as the zero level of a level–set function is a classical way to compute its position in the XFEM litterature. Most of the time, the level–set function is computed as the solution of a PDE [145].

In the present study, in order to locate more precisely the interface, we compute the exact values of the level–set function where this function is the signed distance to a polygon approaching the structure. This is done with the use of a cloud of points that are located on the actual position of the interface. Those points are the vertices of a polygon approximating the structure and, for each node of the mesh, the distance to this polygon is computed (see Fig. 3.18).

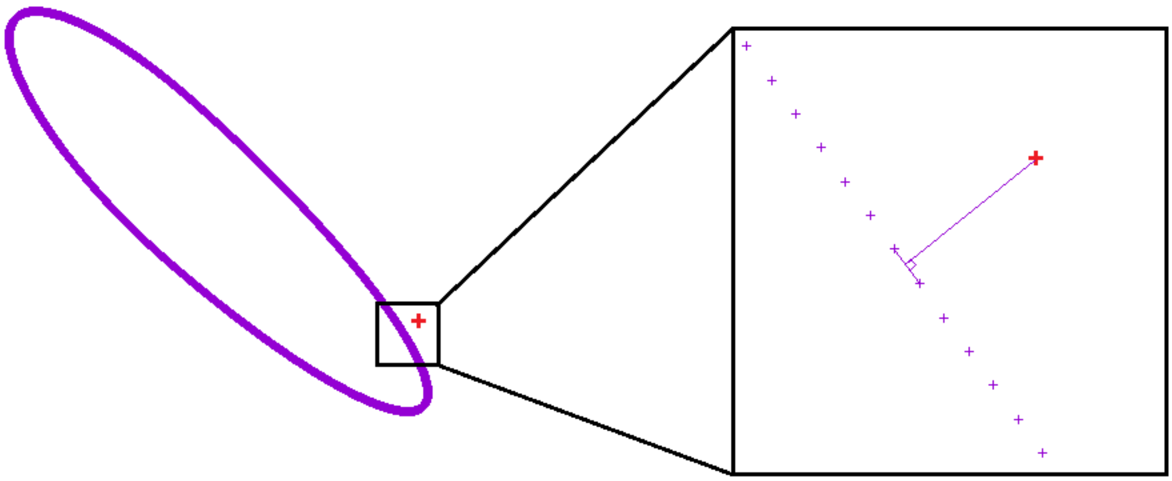


FIGURE 3.18 – The set of points representing the interface and the method to compute the distance function.

In order to reduce drastically the computational cost of this method, at each time step, we compute the distance to the level–set only for the mesh nodes that are needed, i.e. the nodes near the interface.

Note that we can choose the number of points representing the interface. Then, we can ensure an accuracy of high order that does not limit the final approximation of the solution. The choice of the number of points used to represent the interface is a balance between the computational cost and the precision of the computation. We can use more points near the leading edge and the trailing edge of the structure to get more precision in these regions.

3.6.3.2 Integration over the cut cells

The position of the interface is found by solving an optimization routine that identifies the zeros of the level–set function. Every cut cells are divided into sub–cells (see Fig. 3.19), all the points that describe the position of the interface are taken as vertices of some of these sub–cells. When integrating over $\mathscr{F}(\theta_1, \theta_2)$, we integrate over all the sub–cells that are in $\mathscr{F}(\theta_1, \theta_2)$.

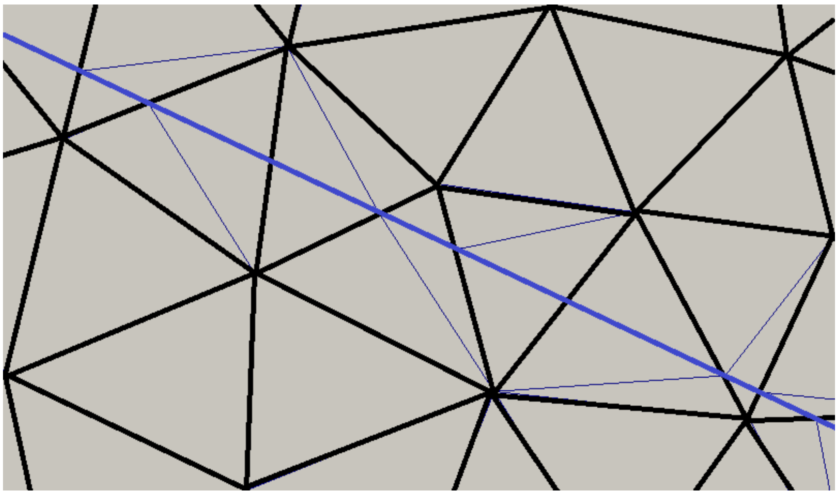


FIGURE 3.19 – Cut elements and sub–cells (used only for integration).

3.6.4 The inverse diffeomorphism

The goal of this section is to show how to compute $f(\mathbf{Y}(\theta_1, \theta_2, .))$ where f is a function defined on $S(0,0)$, for instance $f = \partial_{\theta_j}\mathbf{X}(\theta_1, \theta_2, .)$. The difficulty comes from the fact that, contrary to $\mathbf{X}$, we do not have any explicit expression for $\mathbf{Y}$.

We select all the degrees of freedom $\mathbf{y}_i$ of the cut elements and structure elements in the configuration $(0,0)$ and we compute $\mathbf{x}_i = \mathbf{X}(\theta_1, \theta_2, y_i)$. The set $\{\mathbf{x}_i\}$ forms a cloud of points covering $S(\theta_1, \theta_2)$. For each $\mathbf{y}_i$, we also compute $f(\mathbf{y}_i)$ and we assign its value to $\mathbf{x}_i$.

We have constructed a list of points $\{\mathbf{x}_i\}$ covering $S(\theta_1, \theta_2)$ assigned with the values of $f(\mathbf{y}_i)$. Note that there is a correspondance between the points $\mathbf{x}_i$ and $\mathbf{y}_i$, we have $\mathbf{x}_i = \mathbf{X}(\theta_1, \theta_2, \mathbf{y}_i)$ and $\mathbf{y}_i = (\mathbf{X}(\theta_1, \theta_2))^{-1}(\mathbf{x}_i)$. The function defined on the list of points $\{\mathbf{x}_i\}$ is then $f(\mathbf{X}(\theta_1, \theta_2)^{-1}(\mathbf{x}_i))$.

Now, for every degree of freedom $\mathbf{x}'_i$ linked with $S(\theta_1, \theta_2)$, we can compute $f \circ (\mathbf{X}(\theta_1, \theta_2, .))^{-1}(\mathbf{x}'_i)$ as a weigthed mean of the values of the closest points $\mathbf{x}_i$. In the sequel, we use barycentric coordinates [60, p.21]. In that way, we have defined an approximation of $f \circ (\mathbf{X}(\theta_1, \theta_2, .))^{-1}$.

3.6.5 Numerical results

We consider the configuration represented in Fig. 3.9 with $\Omega = (-1.0, 8.0) \times (y_{\min}, y_{\max})$ where $y_{\min} = -2.4$, $y_{\max} = 2.1$ and where the structure domain is given by (3.41) and (3.42). The initial position of the structure is $(\xi_1, \xi_2) = (-25°, 0)$, the initial inflow boundary datum is given by the Poiseuille profile

$$\mathbf{u}^i(x_2) = \frac{6U_{\mathrm{m}}}{(y_{\max} - y_{\min})^2}(-x_2^2 + (y_{\max} + y_{\min})x_2 - y_{\max}y_{\min}),$$

where U_{m} is the mean speed of the inflow datum. The Reynolds number $\mathcal{R}e = \dfrac{cU_{\mathrm{m}}}{\nu}$, where $c = 1$ is the chord of the profile, is taken as $\mathcal{R}e = 120$. In the sequel, we use $U_{\mathrm{m}} = 1$ and $\nu = \frac{1}{120}$. The initial state of the fluid is computed as the stationary state associated to the datum $\mathbf{u}^i$.

The numerical computations are led on a triangular mesh of 35731 cells locally refined near the boundary, near the structure and near the wake of the structure (see Fig. 3.20). We use the finite element spaces and the time stepping process defined above. The total number of degrees of freedom is equal to 153880 at the initial time and varies according to the number of elements that are discarded.

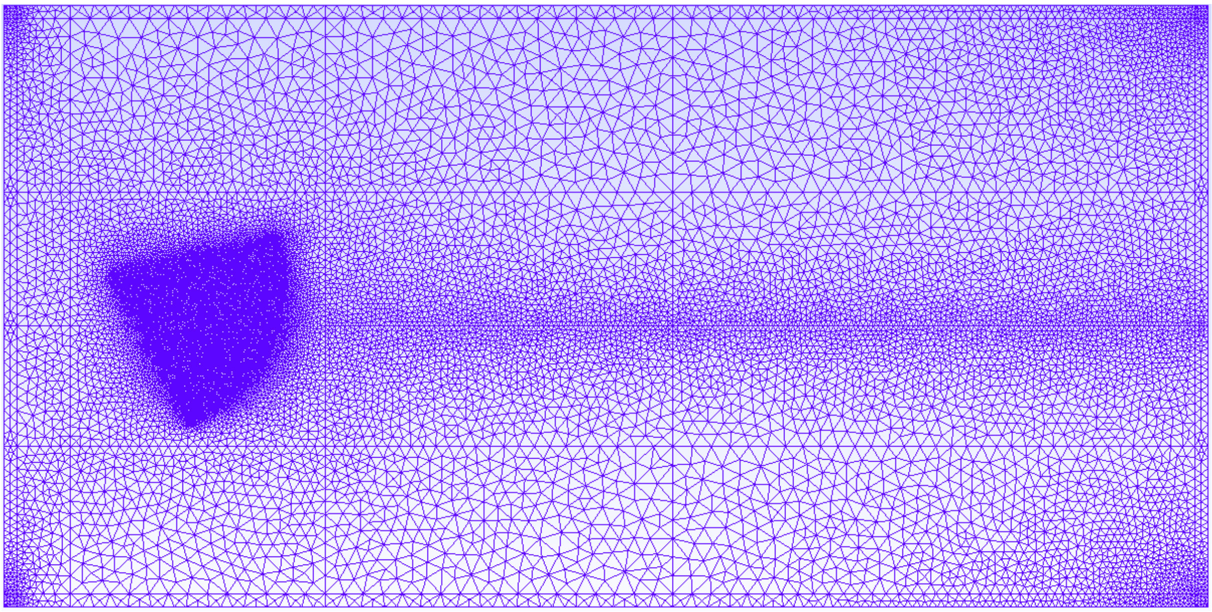


FIGURE 3.20 – The mesh used for the fictitious domain approach.

We consider a boundary perturbation of the form

$$\mathbf{u}^p(t, x_2) = \beta_p z_p(x_2)(\sigma_F(\widetilde{\mathbf{v}}, \widetilde{q})\mathbf{n}_s \cdot \mathbf{n}_s, 0)^T e^{-(t-0.5)^2},$$

where $\widetilde{\mathbf{v}}$ and $\widetilde{q}$ are respectively the velocity and the pressure associated to the most unstable eigenvector of the adjoint problem, $z_p(x_2)$ is a smoothing function defined by

$$z_p(x_2) = F\left(\frac{x_2 - y_{\mathrm{min}}}{(y_{\mathrm{max}} - y_{\mathrm{min}})f}\right) - F\left(1 - \frac{y_{\mathrm{max}} - x_2}{(y_{\mathrm{max}} - y_{\mathrm{min}})f}\right),$$

where

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ (6x^2 - 15x + 10)x^3 & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1, \end{cases}$$

and $f = 0.125$ is a parameter.

Moreover, $\beta_p > 0$ is the maximum value of the perturbation, in the sequel, we use $\beta_p = 0.2$. Since, the maximum value of the inflow $\mathbf{f}_{\mathscr{F}}$ is 1.6, the perturbation size worths 12.5% of the inflow. This perturbation corresponds to one of the most destabilizing normal boundary perturbation, see Fig. 3.21.

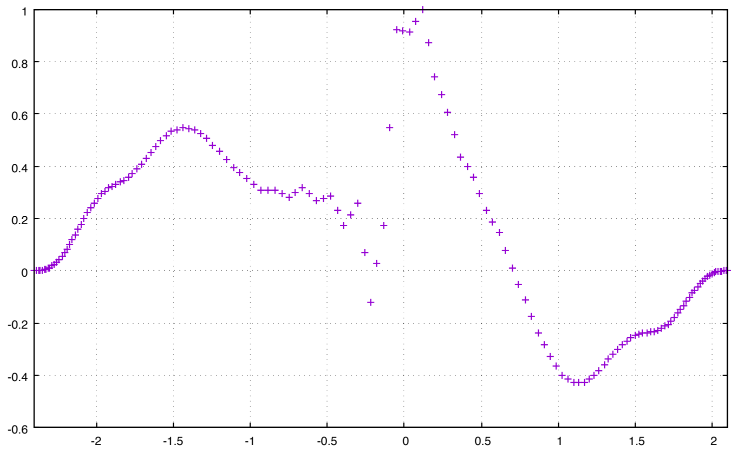


FIGURE 3.21 – The perturbation profile $\sigma_F(\tilde{\mathbf{v}}, \tilde{q})\mathbf{n}_s \cdot \mathbf{n}_s$.

The parameters of the structure are taken as $\rho = 5$ and $k = 12$. We also use the coefficients $\alpha_{\min} = 8.10^{-3}$ and $\gamma_0 = 5.10^{-2}$ (defined respectively in Assumption A and (3.49)).

The time step is given by the CFL condition (3.53) with a coefficient $cfl = 0.8$ and a maximum time step $\Delta t_{\max} = 5.10^{-4}s$.

We run a first simulation in open loop (without any control). The evolution of the magnitude of the velocity is represented in Fig. 3.22.

We also run a simulation in closed loop (with the feedback control). In that way, we can compare the results with and without the control, and assess the efficiency of the feedback control. The results are reported in Fig. 3.23.

In Fig. 3.23a and Fig. 3.23b, we observe that the structure is much more mobile in closed loop. This is understandable because the feedback control tries to stabilize the fluid flow by acting through the structure. The fact that the structure moves shows that the control is active.

In Fig. 3.23c, we represent the difference between the state of the fluid and its stationary solution $\mathbf{v}_h$ defined in (3.52). It has, at first, the same behaviour in closed and open loops, it is even better in closed loop near $t = 1s$. However, after $t = 1.5s$, we see that the control do not manage to stabilize the fluid. Its action perturbs even more the state of the fluid.

If we plot the magnitude of the velocity for the closed loop system, the figures are really close from the one of Fig. 3.22. The instabilities in the wake of the structure are not stabilized by the feedback control.

We use a linear control to stabilize a nonlinear system. Hence, we can obtain the stabilization of the system only for small perturbations. In order to control the flow in a better way, we consider a smaller perturbation ($\beta_p = 0.005$). We finally get the following results (see Fig. 3.24).

We observe that

— the control seems to be efficient when the perturbation reaches the structure,

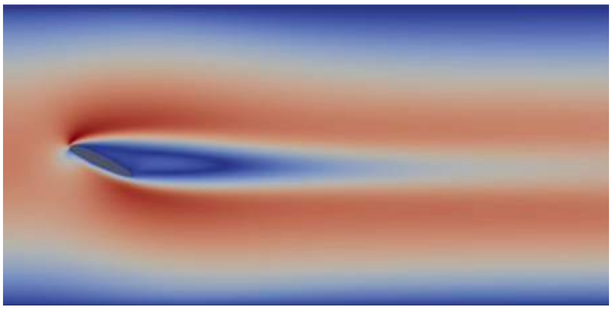
— the control does not stabilize the flow.

Several perspectives can be considered to improve this situation. We can

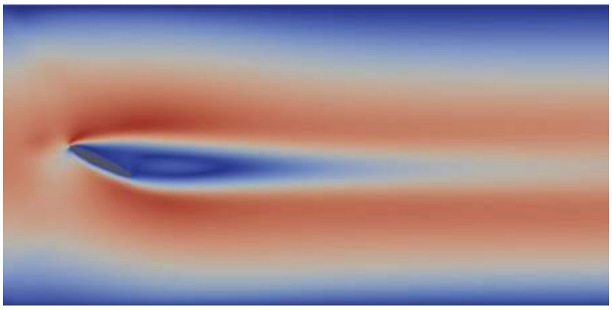
— take higher value for the shift ω in (3.39) (considering $\omega = 10$ instead of $\omega = 6$),

— take an invariant subspace of higher dimension, i.e. increase δ in (3.30),

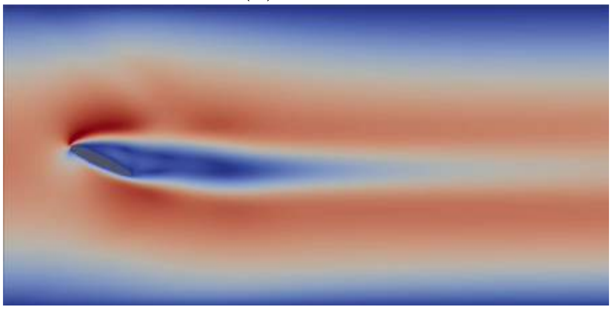
— decrease even more the amplitude of the perturbation β_p.



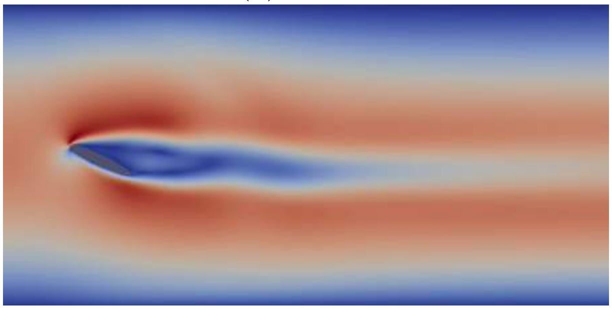
(a) $t = 0s$.



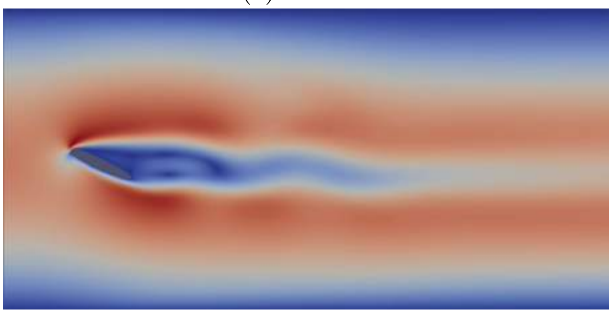
(b) $t = 1s$.



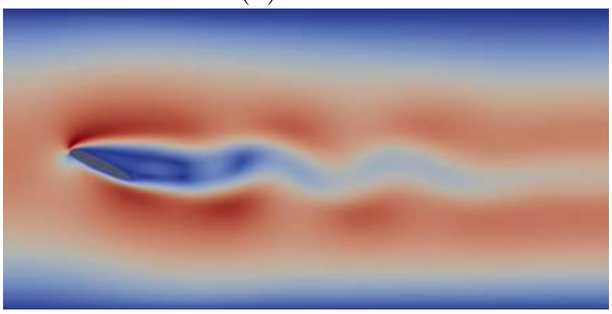
(c) $t = 2s$.



(d) $t = 3s$.

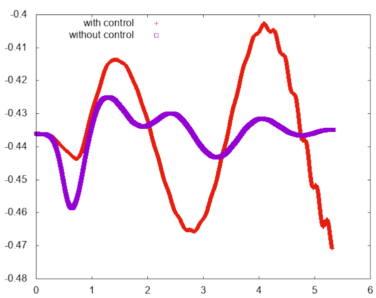


(e) $t = 4s$.

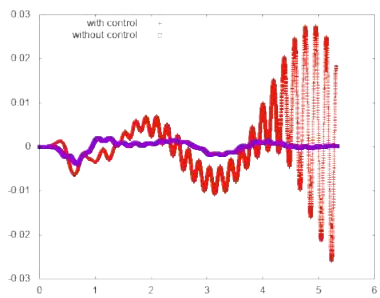


(f) $t = 5s$.

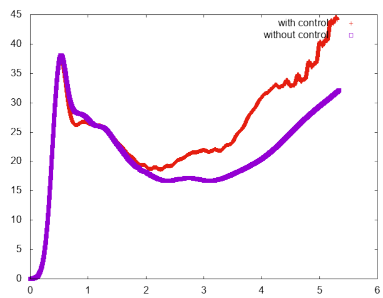
FIGURE 3.22 – The open loop solution ($\mathcal{R}e = 120$).



(a) The evolution of θ_1.

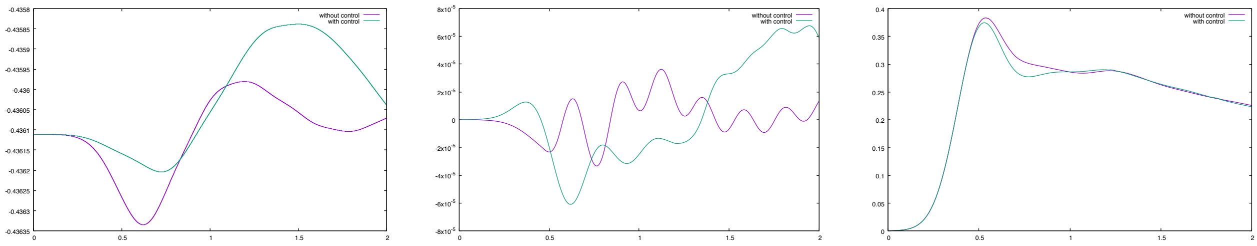


(b) The evolution of θ_2.



(c) The $\mathbf{L}^2$–norm of $\mathbf{v}_h$.

FIGURE 3.23 – Evolution of the numerical system in open (blue) and closed (red) loops ($\beta_p = 0.2$).



(a) The evolution of θ_1. (b) The evolution of θ_2. (c) The $\mathbf{L}^2$–norm of $\mathbf{v}_h$.

FIGURE 3.24 – Evolution of the numerical system in open (blue) and closed (red) loops ($\beta_p = 0.005$).

Annexe A

Proof of Lemma 1.3.3

This section is devoted to the proof of Lemma 1.3.3. We start with some intermediate lemmas that will be used to decompose the intricate terms of Lemma 1.3.3 in smaller pieces.

A.1 Technical Lemmas

The following lemma contains Lipschitz estimates on several terms.

Lemma A.1.1. *For $R > 0$, there exists a constant $C = C(R) > 0$ such that for every $T \in (0, T_0)$ and every $(.,.,\theta_1^j, \theta_2^j) \in \mathbb{B}_R(T)$, the following estimates hold*

$$\|\mathbf{\Phi^0}(\theta_1^a, \theta_2^a) - \mathbf{\Phi^0}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.1}$$

$$\|\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a, \theta_2^a) - \mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.2}$$

$$\|\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a) - \mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.3}$$

$$\|(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi^0}(\theta_1^a, \theta_2^a) - (\partial_{x_j}\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi^0}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.4}$$

$$\|(\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi^0}(\theta_1^a, \theta_2^a) - (\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi^0}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.5}$$

$$\|\mathcal{M}_{\theta_1^a, \theta_2^a} - \mathcal{M}_{\theta_1^b, \theta_2^b}\|_{\mathrm{L}^\infty(0,T)} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.6}$$

$$\|\mathbf{n}_{\theta_1^a, \theta_2^a}(\mathbf{\Phi^0}(\theta_1^a, \theta_2^a)) - \mathbf{n}_{\theta_1^b, \theta_2^b}(\mathbf{\Phi^0}(\theta_1^b, \theta_2^b))\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\partial S_0))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.7}$$

$$\|\det(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a)) - \det(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b, \theta_2^b))\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.8}$$

$$\|\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^a, \theta_2^a, .) - \partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b, \theta_2^b, .)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.9}$$

$$\|\partial_{\theta_k\theta_j}\mathbf{\Phi^0}(\theta_1^a, \theta_2^a, .) - \partial_{\theta_k\theta_j}\mathbf{\Phi^0}(\theta_1^b, \theta_2^b, .)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.10}$$

$$\|\,|\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a, \theta_2^a)\mathbf{t}_0| - |\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^b, \theta_2^b)\mathbf{t}_0|\,\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\partial S_0))} \leq CT\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.11}$$

and

$$\|\partial_t\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a, \theta_2^a) - \partial_t\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq C\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.12}$$

$$\|\partial_t(\mathbf{\Psi^0}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi^0}(\theta_1^a, \theta_2^a) - \partial_t(\mathbf{\Psi^0}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi^0}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C\|\theta^a - \theta^b\|_{\Theta_T}, \tag{A.13}$$

$$\|\partial_t(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a)) \circ \mathbf{\Phi^0}(\theta_1^a, \theta_2^a) - \partial_t(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b, \theta_2^b)) \circ \mathbf{\Phi^0}(\theta_1^b, \theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C\|\theta^a - \theta^b\|_{\Theta_T}. \tag{A.14}$$

Moreover, for every $(\tilde{\mathbf{u}}^j, ., \theta_1^j, \theta_2^j) \in \mathbb{B}_R(T)$, the following estimates hold on $\mathcal{G}$ defined in (1.67)

$$\|\mathcal{G}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a) - \mathcal{G}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b)\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\partial S_0))} \leq C(\|\theta^a - \theta^b\|_{\Theta_T} + \|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T}), \tag{A.15}$$

$$\|\nabla\tilde{\mathbf{u}}^a - \mathcal{G}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a) - \nabla\tilde{\mathbf{u}}^b + \mathcal{G}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b)\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\partial S_0))} \leq CT(\|\theta^a - \theta^b\|_{\Theta_T} + \|\tilde{\mathbf{u}}^a - \tilde{\mathbf{u}}^b\|_{\mathbb{U}_T}). \tag{A.16}$$

In particular, as a direct application of Lemma A.1.1, using that $(0,0,0,0) \in \mathbb{B}_R(T)$, we obtain the following lemma.

Lemma A.1.2. *For $R > 0$, there exists a constant $C = C(R) > 0$, such that for every $T \in (0, T_0)$ and every $(.,.,\theta_1,\theta_2) \in \mathbb{B}_R(T)$, the following estimates hold*

$$\|\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1,\theta_2) - \mathrm{I}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT, \tag{A.17}$$

$$\|\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1,\theta_2,\mathbf{\Phi^0}(\theta_1,\theta_2)) - \mathrm{I}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq CT, \tag{A.18}$$

$$\|\partial_{x_j}\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1,\theta_2) \circ \mathbf{\Phi^0}(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\Omega))} \leq CT, \tag{A.19}$$

$$\|\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1,\theta_2) \circ \mathbf{\Phi^0}(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq CT, \tag{A.20}$$

$$\|\mathcal{M}_{\theta_1,\theta_2} - \mathcal{M}_{0,0}\|_{\mathrm{L}^\infty(0,T)} \leq CT, \tag{A.21}$$

$$\|\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi^0}(\theta_1,\theta_2)) - \mathbf{n}_0\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\partial S_0))} \leq CT, \tag{A.22}$$

$$\|\,|\mathcal{J}_{\mathbf{\Phi^0}}\mathbf{t}_0| - 1\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\partial S_0))} \leq CT, \tag{A.23}$$

and

$$\|\partial_t(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1,\theta_2))\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\Omega))} \leq C, \tag{A.24}$$

$$\left\|\frac{\partial}{\partial t}(\mathbf{\Psi^0}(\theta_1,\theta_2)) \circ \mathbf{\Phi^0}(\theta_1,\theta_2)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C, \tag{A.25}$$

$$\left\|\frac{\partial}{\partial t}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1,\theta_2)) \circ \mathbf{\Phi^0}(\theta_1,\theta_2)\right\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\Omega))} \leq C. \tag{A.26}$$

Moreover, for every $(\tilde{\mathbf{u}},.,\theta_1,\theta_2) \in \mathbb{B}_R(T)$, we have the following estimate on $\mathcal{G}$

$$\|\nabla\tilde{\mathbf{u}} - \mathcal{G}(\theta_1,\theta_2,\tilde{\mathbf{u}})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\partial S_0))} \leq CT. \tag{A.27}$$

Proof of Lemma A.1.1. Three kinds of estimates have to be proven. First estimates (A.1)–(A.10) are of the type

$$\|\alpha(\theta_1^a,\theta_2^a) - \alpha(\theta_1^b,\theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbb{X})} \leq CT\|(\theta_1^a,\theta_2^a) - (\theta_1^b,\theta_2^b)\|_{\Theta_T},$$

where α is a differentiable function defined on $\mathbb{D}_\Theta$ and valued in $\mathbb{X}$. We thus use Taylor series and get

$$\|\alpha(\theta_1^a,\theta_2^a) - \alpha(\theta_1^b,\theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbb{X})} \leq \sup_{(\theta_1,\theta_2)\in\mathbb{D}_\Theta} \|\nabla_\theta\alpha(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,T;\mathbb{X})}\|\theta^a - \theta^b\|_{\mathrm{L}^\infty(0,T)}.$$

According to the definition of $\mathbb{B}_R(T)$ in (1.70), $\theta^a(0) = \theta^b(0) = (0,0)$, we finish with

$$\|\theta^a - \theta^b\|_{\mathrm{L}^\infty(0,T)} \leq T\|\theta^a - \theta^b\|_{\Theta_T}.$$

The second type of estimates (A.12)–(A.14) is of the form

$$\|\alpha(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \alpha(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbb{X})} \leq C\|(\theta_1^a,\theta_2^a) - (\theta_1^b,\theta_2^b)\|_{\Theta_T},$$

where α is now a function defined on $\mathbb{D}_\Theta \times \mathbb{R}^2$ with values in $\mathbb{X}$. We use the same strategy and get

$$\begin{aligned}&\|\alpha(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \alpha(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{\mathrm{L}^\infty(0,T;\mathbb{X})}\\ &\qquad \leq \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta\\ |\omega_1|+|\omega_2|\leq R}}\|\nabla_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{\mathrm{L}^\infty(0,T;\mathbb{X})}(\|\theta^a-\theta^b\|_{\mathrm{L}^\infty(0,T)}+\|\dot\theta^a-\dot\theta^b\|_{\mathrm{L}^\infty(0,T)}).\end{aligned}$$

Note that contrary to the first type of estimates, we do not have the decay in T because we did not enforce $\dot\theta^a(0) = \dot\theta^b(0)$.

Estimate (A.15) is a direct consequence of (A.16). The last estimate to prove is (A.16). We do it via the computation

$$\begin{aligned}
(\mathcal{G}(\theta_1^a, \theta_2^a, \tilde{\mathbf{u}}^a)-&\mathcal{G}(\theta_1^b, \theta_2^b, \tilde{\mathbf{u}}^b) - \nabla\tilde{\mathbf{u}}^a + \nabla\tilde{\mathbf{u}}^b)_{ij} \\
&= \sum_k \left(\mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a, .)\circ\mathbf{\Phi}^a)_{ki} - \mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b, .)\circ\mathbf{\Phi}^b)_{ki}\right)\tilde{u}_k^a \\
&+ \sum_k \mathrm{cof}(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b, .)\circ\mathbf{\Phi}^b)_{ki}(\tilde{u}_k^a - \tilde{u}_k^b) \\
&+ \sum_{k,l} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^a, \theta_2^a, \mathbf{\Phi}^a) - \mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b, \mathbf{\Phi}^b))_{ki}\frac{\partial\tilde{u}_k^a}{\partial y_l}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a, \theta_2^a, \mathbf{\Phi}^a) \\
&+ \sum_{k,l}\left(\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b, \mathbf{\Phi}^b))_{ki}\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a, \theta_2^a, \mathbf{\Phi}^a) - \delta_{ki}\delta_{lj}\right)\left(\frac{\partial\tilde{u}_k^a}{\partial y_l} - \frac{\partial\tilde{u}_k^b}{\partial y_l}\right) \\
&+ \sum_{k,l} \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi}}(\theta_1^b, \theta_2^b, \mathbf{\Phi}^b))_{ki}\frac{\partial\tilde{u}_k^b}{\partial y_l}\left(\frac{\partial\Psi_l}{\partial x_j}(\theta_1^a, \theta_2^a, \mathbf{\Phi}^a) - \frac{\partial\Psi_l}{\partial x_j}(\theta_1^b, \theta_2^b, \mathbf{\Phi}^b)\right),
\end{aligned}$$

and with the use of estimates (A.3), (A.4), (A.18) and (A.19) we get estimate (A.16). □

A.2 Detailed proof of Lemma 1.3.3

Proof. In all the estimates we use Lemmas A.1.1 and A.1.2.

• **Estimate** (1.71) **:** using (A.18) and (A.3), we get

$$\begin{aligned}
\|\mathbf{F}^1(\theta_1^a, \theta_2^a, \mathbf{v}^a) -&\mathbf{F}^1(\theta_1^b, \theta_2^b, \mathbf{v}^b)\|_{\mathbb{F}_T} \\
&\leq \left\|\mathrm{I} - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a))^T\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\left\|\frac{\partial\mathbf{v}^a}{\partial t} - \frac{\partial\mathbf{v}^b}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{L}^2)} \\
&\quad + \left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b, \theta_2^b))^T - \mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a, \theta_2^a))^T\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\left\|\frac{\partial\mathbf{v}^b}{\partial t}\right\|_{\mathrm{L}^2(\mathbf{L}^2)} \\
&\leq KT(\|\theta^a - \theta^b\|_{\Theta_T} + \|\mathbf{v}^a - \mathbf{v}^b\|_{\mathbb{U}_T}).
\end{aligned}$$

Now, using (A.26), (A.25), (A.14), (A.13), (A.3) and the estimate

$$\|\mathbf{v}\|_{\mathrm{L}^2(0,T;\mathbf{H}^1(\mathscr{F}_0))} \leq T^{1/2}\|\mathbf{v}\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^1(\mathscr{F}_0))},$$

we obtain

$$
\begin{aligned}
&\|\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\\
&\qquad\le \|\mathrm{cof}(\partial_t\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi^0}(\theta_1^b,\theta_2^b)-\partial_t\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi^0}(\theta_1^a,\theta_2^a))^T\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{L}^2)}\\
&\qquad\quad+\|\mathrm{cof}(\partial_t\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi^0}(\theta_1^b,\theta_2^b))^T\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^b-\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{L}^2)}\\
&\qquad\quad+\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi^0}(\theta_1^b,\theta_2^b)-\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi^0}(\theta_1^a,\theta_2^a))^T\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\qquad\qquad\qquad\qquad\|\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{H}^1)}\|\partial_t\mathbf{\Psi^0}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi^0}\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\qquad\quad+\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi^0}(\theta_1^b,\theta_2^b))^T\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^b-\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{H}^1)}\|\partial_t\mathbf{\Psi^0}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi^0}\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\qquad\quad+\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi^0}(\theta_1^b,\theta_2^b))^T\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathbf{v}^b\|_{\mathrm{L}^2(\mathbf{H}^1)}\\
&\qquad\qquad\qquad\qquad\|\partial_t(\mathbf{\Psi^0}(\theta_1^b,\theta_2^b))\circ\mathbf{\Phi^0}-\partial_t(\mathbf{\Psi^0}(\theta_1^a,\theta_2^a))\circ\mathbf{\Phi^0}\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\qquad\le KT^{1/2}(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_T}+\|\theta^a-\theta^b\|_{\Theta_T}).
\end{aligned}
$$

In the following estimate we use the Sobolev embedding : $\mathbf{H}^{1/2+\varepsilon_0}\hookrightarrow\mathbf{L}^4$ (see [1, Theorem 7.58]). We also use the fact that $\mathcal{J}_{\mathbf{\Psi^0}}$ is the identity near the boundary, i.e. $\mathcal{E}$ has support in Ω_ε (defined in Lemma 1.1.4), then $\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1,\theta_2,\mathbf{\Phi^0}(\theta_1,\theta_2,\mathbf{y}))-\mathrm{I}=0$ for $\mathbf{y}\in\Omega\backslash\Omega_\varepsilon$. Hence,

$$
\begin{aligned}
\left\|\frac{\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}\dfrac{\partial\Psi_l^0}{\partial x_j}\dfrac{\partial\Psi_m^0}{\partial x_j}-\delta_{ki}\delta_{lj}\delta_{mj}}{\prod\limits_j r_j^\beta}\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty(\Omega))}
&\le\left\|\frac{\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}\dfrac{\partial\Psi_l^0}{\partial x_j}\dfrac{\partial\Psi_m^0}{\partial x_j}-\delta_{ki}\delta_{lj}\delta_{mj}}{\prod\limits_j r_j^\beta}\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty(\Omega_\varepsilon))}\\
&\le\left\|\frac{1}{\prod\limits_j r_j^\beta}\right\|_{\mathbf{L}^\infty(\Omega_\varepsilon)}\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}\frac{\partial\Psi_l^0}{\partial x_j}\frac{\partial\Psi_m^0}{\partial x_j}-\delta_{ki}\delta_{lj}\delta_{mj}\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty(\Omega))}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\|\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\\
&\leq \nu\sum_{j,k,l,m}\left\|\frac{\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a))_{ki}\dfrac{\partial\Psi_l^0}{\partial x_j}(\theta_1^a,\theta_2^a)\dfrac{\partial\Psi_m^0}{\partial x_j}(\theta_1^a,\theta_2^a)-\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))_{ki}\dfrac{\partial\Psi_l^0}{\partial x_j}(\theta_1^b,\theta_2^b)\dfrac{\partial\Psi_m^0}{\partial x_j}(\theta_1^b,\theta_2^b)}{\prod_n r_n^\beta}\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\times\left\|\frac{\partial^2 v_k^a}{\partial y_l\partial y_m}\right\|_{\mathrm{L}^2(\mathbf{L}^2_\beta)}+\left\|\frac{\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))\dfrac{\partial\Psi_l^0}{\partial x_j}(\theta_1^b,\theta_2^b)\dfrac{\partial\Psi_m^0}{\partial x_j}(\theta_1^b,\theta_2^b)-\delta_{lm}\delta_{ki}\delta_{li}}{\prod_n r_n^\beta}\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\left\|\frac{\partial^2 v_k^a}{\partial y_l\partial y_m}-\frac{\partial^2 v_k^b}{\partial y_l\partial y_m}\right\|_{\mathrm{L}^2(\mathbf{L}^2_\beta)}\\
&+2\nu\sum_{j,k,l}\left\|\frac{\partial}{\partial x_j}(\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a))_{ki})\frac{\partial\Psi_l^0}{\partial x_j}(\theta_1^a,\theta_2^a)-\frac{\partial}{\partial x_j}(\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))_{ki})\frac{\partial\Psi_l^0}{\partial x_j}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^4)}\|\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{W}^{1,4})}\\
&\qquad+\left\|\frac{\partial}{\partial x_j}\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))_{ki}\frac{\partial\Psi_l^0}{\partial x_j}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^4)}\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathrm{L}^2(\mathbf{W}^{1,4})}\\
&+\nu\sum_{j,k,l}\left\|\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0})_{ki}(\theta_1^a,\theta_2^a)\frac{\partial^2\Psi_l^0}{\partial x_j^2}(\theta_1^a,\theta_2^a)-\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial^2\Psi_l^0}{\partial x_j^2}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^4)}\|\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{W}^{1,4})}\\
&\qquad+\left\|\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial^2\Psi_l^0}{\partial x_j^2}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^4)}\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathrm{L}^2(\mathbf{W}^{1,4})}\\
&+\nu\sum_{j,k}\left\|\frac{\partial^2}{\partial x_j^2}\operatorname{cof}\Big(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)-\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b)\Big)_{ki}\right\|_{\mathrm{L}^\infty(\mathbf{L}^2)}\|\mathbf{v}^a\|_{\mathrm{L}^2(\mathbf{L}^\infty)}+\left\|\frac{\partial^2}{\partial x_j^2}\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))_{ki}\right\|_{\mathrm{L}^\infty(\mathbf{L}^2)}\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathrm{L}^2(\mathbf{L}^\infty)}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\left\|\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a))_{ki}\frac{\partial\Psi_l^0}{\partial x_j}(\theta_1^a,\theta_2^a)\frac{\partial\Psi_m^0}{\partial x_j}(\theta_1^a,\theta_2^a)-\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))_{ki}\frac{\partial\Psi_l^0}{\partial x_j}(\theta_1^b,\theta_2^b)\frac{\partial\Psi_m^0}{\partial x_j}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\quad\leq\|\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)-\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))_{ki}\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\qquad\qquad+\|\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)-\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\qquad\qquad+\|\operatorname{cof}(\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b))\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\|\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^a,\theta_2^a)-\mathcal{J}_{\mathbf{\Psi}^0}(\theta_1^b,\theta_2^b)\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\\
&\quad\leq KT\|\theta^a-\theta^b\|_{\Theta_T},
\end{aligned}
$$

and with similar estimates, we get

$$\|\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\leq KT(\|\theta^a-\theta^b\|_{\Theta_T}+\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_T}).$$

The estimate on $\mathbf{F}^4$ can be obtained by the decomposition

$$\begin{aligned}
&\|\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\\
&\leq \sum_{j,k,r}\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}(\theta_1^a,\theta_2^a)\frac{\partial}{\partial x_j}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}(\theta_1^a,\theta_2^a)-\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial}{\partial x_j}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathrm{L}^\infty)}\|v_k^a v_r^a\|_{\mathrm{L}^2(\mathrm{L}^2)}\\
&\qquad+\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Psi^0}})_{ki}(\theta_1^b,\theta_2^b)\frac{\partial}{\partial x_j}(\mathcal{J}_{\mathbf{\Psi^0}})_{ri}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathrm{L}^\infty)}\|v_k^a v_r^a-v_k^b v_r^b\|_{\mathrm{L}^2(\mathrm{L}^2)}\\
&\qquad+\sum_{k,r}\left\|\det(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^a,\theta_2^a))^2\frac{\partial\mathbf{\Phi^0}}{\partial y_r}(\theta_1^a,\theta_2^a)-\det(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b))^2\frac{\partial\mathbf{\Phi^0}}{\partial y_r}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\left\|v_k^a\frac{\partial v_r^a}{\partial y_k}\right\|_{\mathrm{L}^2(\mathrm{L}^2)}\\
&\qquad+\left\|\det(\mathcal{J}_{\mathbf{\Psi^0}}(\theta_1^b,\theta_2^b))^2\frac{\partial\mathbf{\Phi^0}}{\partial y_r}(\theta_1^b,\theta_2^b)\right\|_{\mathrm{L}^\infty(\mathbf{L}^\infty)}\left\|v_k^a\frac{\partial v_r^a}{\partial y_k}-v_k^b\frac{\partial v_r^b}{\partial y_k}\right\|_{\mathrm{L}^2(\mathrm{L}^2)}.
\end{aligned}$$

At this point we use estimates (A.17), (A.18), (A.3), (A.2), (A.4), (A.19), (A.8) and the Sobolev embedding $\mathbf{H}^{1/2+\varepsilon_0}\hookrightarrow\mathbf{L}^4$ to obtain

$$\|\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\leq C\left(T\|\theta^a-\theta^b\|_{\Theta_T}+\|v_k^a v_r^a-v_k^b v_r^b\|_{\mathrm{L}^2(\mathrm{L}^2)}+\left\|v_k^a\frac{\partial v_r^a}{\partial y_k}-v_k^b\frac{\partial v_r^b}{\partial y_k}\right\|_{\mathrm{L}^2(\mathrm{L}^2)}\right).$$

Hölder inequalities yield

$$\begin{aligned}
\left\|v_k^a\frac{\partial v_r^a}{\partial y_k}-v_k^b\frac{\partial v_r^b}{\partial y_k}\right\|_{\mathrm{L}^2(\mathrm{L}^2)}&\leq\left\|(v_k^a-v_k^b)\frac{\partial v_r^a}{\partial y_k}\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathscr{F}_0))}+\left\|v_k^b\left(\frac{\partial v_r^a}{\partial y_k}-\frac{\partial v_r^b}{\partial y_k}\right)\right\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathscr{F}_0))}\\
&\leq T^{1/4}\left(\left\|(v_k^a-v_k^b)\frac{\partial v_r^a}{\partial y_k}\right\|_{\mathrm{L}^4(0,T;\mathrm{L}^2(\mathscr{F}_0))}+\left\|v_k^b\left(\frac{\partial v_r^a}{\partial y_k}-\frac{\partial v_r^b}{\partial y_k}\right)\right\|_{\mathrm{L}^4(0,T;\mathrm{L}^2(\mathscr{F}_0))}\right)\\
&\leq CT^{1/4}\left(\left\|v_k^a-v_k^b\right\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^{10})}\left\|\frac{\partial v_r^a}{\partial y_k}\right\|_{\mathrm{L}^4(0,T;\mathrm{L}^{5/2})}+\|v_k^b\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^{10})}\left\|\frac{\partial v_r^a}{\partial y_k}-\frac{\partial v_r^b}{\partial y_k}\right\|_{\mathrm{L}^4(0,T;\mathrm{L}^{5/2})}\right).
\end{aligned}$$

To estimate the previous terms, we adapt the proof of [34, p. 298]. We use the Sobolev interpolation

$$\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/4}(\mathscr{F}_0)}\leq C\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/2}(\mathscr{F}_0)}^{1/2}\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{L}^2(\mathscr{F}_0)}^{1/2},$$

and we compute

$$\begin{aligned}
\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathrm{L}^4(0,T;\mathbf{H}^{1/4}(\mathscr{F}_0))}^4&=\int_0^T\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/4}(\mathscr{F}_0)}^4\mathrm{d}t\leq C^4\int_0^T\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{H}^{1/2}(\mathscr{F}_0)}^2\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathbf{L}^2(\mathscr{F}_0)}^2\mathrm{d}t\\
&\leq C^4\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathrm{L}^2(0,T;\mathbf{H}^{1/2}(\mathscr{F}_0))}^2\left\|\frac{\partial v_r}{\partial y_k}\right\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\mathscr{F}_0))}^2.
\end{aligned}$$

The same technique can be used on the term $\|v_k^a v_r^a-v_k^b v_r^b\|_{\mathrm{L}^2(\mathrm{L}^2)}$. Then the Sobolev embeddings $\mathbf{H}^{1/4}(\mathscr{F}_0)\hookrightarrow\mathbf{L}^{5/2}(\mathscr{F}_0)$ and $\mathbf{H}^1(\mathscr{F}_0)\hookrightarrow\mathbf{L}^{10}(\mathscr{F}_0)$ yield the estimate

$$\|\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\|_{\mathbb{F}_T}\leq CT^{1/4}(\|\theta^a-\theta^b\|_{\Theta_T}+\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_T}).$$

The following estimate uses (A.18) and (A.14),

$$\begin{aligned}
&\|\mathbf{F}^5(\theta_1^a,\theta_2^a,q^a)-\mathbf{F}^5(\theta_1^b,\theta_2^b,q^b)\|_{\mathbb{F}_T}\\
&\qquad\leq\left\|\frac{\mathcal{J}_{\mathbf{\Psi^0}}^T(\theta_1^b,\theta_2^b)-\mathcal{J}_{\mathbf{\Psi^0}}^T(\theta_1^a,\theta_2^a)}{\prod_j r_j^\beta}\right\|_{\mathbf{L}^\infty}\|q^a\|_{\mathrm{L}^2(\mathrm{H}^1_\beta)}+\left\|\frac{\mathrm{I}-\mathcal{J}_{\mathbf{\Psi^0}}^T(\theta_1^b,\theta_2^b)}{\prod_j r_j^\beta}\right\|_{\mathrm{L}^\infty}\|q^a-q^b\|_{\mathrm{L}^2(\mathrm{H}^1_\beta)}\\
&\qquad\leq KT(\|\theta^a-\theta^b\|_{\Theta_T}+\|q^a-q^b\|_{\mathbb{P}_T}).
\end{aligned}$$

• **Estimate** (1.72) **:** we use the fact that $\mathbf{H}^2(\mathscr{F}_0)$ is an algebra and estimates (A.17), (A.2), (A.9), (A.24), (A.12) and (A.10),

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}\left(\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\right)\right\|_{\mathrm{L}^2(0,T;\mathbf{H}^{3/2}(\partial S_0))}\\
&\leq \sum_{j=1}^{2}\|\ddot\theta_j^a-\ddot\theta_j^b\|_{\mathrm{L}^2(0,T)}\left\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a,\theta_2^a))^T\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})-\partial_{\theta_j}\mathbf{\Phi^0}(0,0,\mathbf{y})\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))}\\
&+\sum_{j=1}^{2}\|\ddot\theta_j^b\|_{\mathrm{L}^2(0,T)}\Big\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})\\
&\qquad\qquad-\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^b,\theta_2^b,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\mathbf{y})\Big\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))}\\
&+\sum_{j=1}^{2}\|\dot\theta_j^a-\dot\theta_j^b\|_{\mathrm{L}^2(0,T)}\left\|\frac{\partial}{\partial t}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a,\theta_2^a))^T\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})\right\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))}\\
&+\sum_{j=1}^{2}\|\dot\theta_j^b\|_{\mathrm{L}^2(0,T)}\Big\|\frac{\partial}{\partial t}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})\\
&\qquad\qquad-\frac{\partial}{\partial t}\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^b,\theta_2^b,\mathbf{y}))^T\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\mathbf{y})\Big\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))}\\
&+\sum_{j,k=1}^{2}\|\dot\theta_j^a\dot\theta_k^a-\dot\theta_j^b\dot\theta_k^b\|_{\mathrm{L}^2(0,T)}\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j\theta_k}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))}\\
&+\sum_{j,k=1}^{2}\|\dot\theta_j^b\dot\theta_k^b\|_{\mathrm{L}^2(0,T)}\|\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^a,\theta_2^a,\mathbf{y}))^T\partial_{\theta_j\theta_k}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})\\
&\qquad\qquad-\mathrm{cof}(\mathcal{J}_{\mathbf{\Phi^0}}(\theta_1^b,\theta_2^b,\mathbf{y}))^T\partial_{\theta_j\theta_k}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\mathbf{y})\|_{\mathrm{L}^\infty(0,T;\mathbf{H}^2(\mathscr{F}_0))},
\end{aligned}$$

and $\|\dot\theta_j^a\dot\theta_k^a-\dot\theta_j^b\dot\theta_k^b\|_{\mathrm{L}^2(0,T;\mathbb{R})}\leq T^{1/2}\|\dot\theta_j^a\dot\theta_k^a-\dot\theta_j^b\dot\theta_k^b\|_{\mathrm{L}^\infty(0,T;\mathbb{R})}$.

We have proven that $\left\|\partial_t\left(\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\right)\right\|_{\mathrm{L}^2(0,T;\mathbf{H}^{3/2}(\partial S_0))}\leq KT^{1/2}\|\theta^a-\theta^b\|_{\Theta_T}$. With the same technique, we also prove $\|\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{\mathrm{L}^2(0,T;\mathbf{H}^{3/2}(\partial S_0))}\leq KT^{1/2}\|\theta^a-\theta^b\|_{\Theta_T}$ and we get estimate (1.72).

• **Estimate** (1.73) **:** we use the following decomposition

$$
\begin{aligned}
&[\mathbf{S}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a,\tilde{p}^a)-\mathbf{S}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b,\tilde{p}^b)]_j\\
&=\left[\left(\mathcal{M}_{\theta_1^b,\theta_2^b}-\mathcal{M}_{\theta_1^a,\theta_2^a}\right)\begin{pmatrix}\ddot{\theta}_1^a\\ \ddot{\theta}_2^a\end{pmatrix}\right]_j+\left[\left(\mathcal{M}_{0,0}-\mathcal{M}_{\theta_1^b,\theta_2^b}\right)\begin{pmatrix}\ddot{\theta}_1^a-\ddot{\theta}_1^b\\ \ddot{\theta}_2^a-\ddot{\theta}_2^b\end{pmatrix}\right]_j\\
&\qquad\qquad+\left[\mathbf{M_I}(\theta_1^a,\theta_2^a,\dot{\theta}_1^a,\dot{\theta}_2^a)-\mathbf{M_I}(\theta_1^b,\theta_2^b,\dot{\theta}_1^b,\dot{\theta}_2^b)\right]_j\\
&+\int_{\partial S_0}|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|\left[\tilde{p}^a\mathrm{I}-\nu(\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)+\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)^T)\right](\mathbf{n}_{\theta_1^a,\theta_2^a}\circ\mathbf{\Phi^0}^a)\\
&\qquad\qquad\cdot(\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\gamma_y)-\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\gamma_y))\\
&+\int_{\partial S_0}|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|\left[\tilde{p}^a\mathrm{I}-\nu(\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)+\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)^T)\right](\mathbf{n}_{\theta_1^a,\theta_2^a}\circ\mathbf{\Phi^0}^a-\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi^0}^b)\cdot\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\gamma_y)\\
&+\int_{\partial S_0}-\nu|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|\Big[\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)+\mathcal{G}(\theta_1^a,\theta_2^a,\tilde{\mathbf{u}}^a)^T-\mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)\\
&\qquad\qquad-\mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)^T-\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)-\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T\Big](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi^0}^b)\cdot\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\gamma_y)\\
&+\int_{\partial S_0}|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|\left[(\tilde{p}^a-\tilde{p}^b)\mathrm{I}-\nu(\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)+\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T)\right](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi^0}^b)\\
&\qquad\qquad\cdot(\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\gamma_y)-\partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y))\\
&+\int_{\partial S_0}|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|\left[(\tilde{p}^a-\tilde{p}^b)\mathrm{I}-\nu(\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)+\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T)\right](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi^0}^b-\mathbf{n}_0)\cdot\partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\\
&+\int_{\partial S_0}(|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|-1)\left[(\tilde{p}^a-\tilde{p}^b)\mathrm{I}-\nu(\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)+\nabla(\tilde{\mathbf{u}}^a-\tilde{\mathbf{u}}^b)^T)\right]\mathbf{n}_0\cdot\partial_{\theta_j}\mathbf{\Phi^0}(0,0,\gamma_y)\\
&+\int_{\partial S_0}(|\mathcal{J}^a_{\mathbf{\Phi^0}}\mathbf{t}_0|-|\mathcal{J}^b_{\mathbf{\Phi^0}}\mathbf{t}_0|)\left[\tilde{p}^b\mathrm{I}-\nu(\mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)+\mathcal{G}(\theta_1^b,\theta_2^b,\tilde{\mathbf{u}}^b)^T)\right](\mathbf{n}_{\theta_1^b,\theta_2^b}\circ\mathbf{\Phi^0}^b)\cdot\partial_{\theta_j}\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\gamma_y),
\end{aligned}
$$

and we use the estimate

$$
\begin{aligned}
&\|\mathbf{M_I}(\theta_1^a,\theta_2^a,\dot{\theta}_1^a,\dot{\theta}_2^a)-\mathbf{M_I}(\theta_1^b,\theta_2^b,\dot{\theta}_1^b,\dot{\theta}_2^b)\|_{\mathrm{L}^2(0,T)}\\
&\qquad\le K(\|(\dot{\theta}_1^a-\dot{\theta}_1^b)^2\|_{\mathrm{L}^4}+\|(\dot{\theta}_1^a-\dot{\theta}_1^b)(\dot{\theta}_2^a-\dot{\theta}_2^b)\|_{\mathrm{L}^2}+\|(\dot{\theta}_2^a-\dot{\theta}_2^b)^2\|_{\mathrm{L}^4})\\
&\qquad\le KT^{1/2}(\|\dot{\theta}_1^a-\dot{\theta}_1^b\|^2_{\mathrm{L}^\infty}+\|(\dot{\theta}_1^a-\dot{\theta}_1^b)(\dot{\theta}_2^a-\dot{\theta}_2^b)\|_{\mathrm{L}^\infty}+\|\dot{\theta}_2^a-\dot{\theta}_2^b\|^2_{\mathrm{L}^\infty}),
\end{aligned}
$$

and (A.9), (A.21), (A.22), (A.6), (A.27), (A.7), (A.11), (A.23) and (A.16) to conclude and obtain (1.73).

• **Estimate** (1.74) **:** we use the Lipschitz regularity of $\mathbf{f}_{\mathscr{F}}$ and estimate (A.1),

$$
\begin{aligned}
&\|\mathbf{f}_{\mathscr{F}}(t,\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y}))-\mathbf{f}_{\mathscr{F}}(t,\mathbf{\Phi^0}(\theta_1^b,\theta_2^b,\mathbf{y}))\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\mathscr{F}_0))}\\
&\qquad\le C\|\mathbf{f}_{\mathscr{F}}\|_{\mathrm{L}^2(0,T;\mathbf{W}^{1,\infty}(\Omega))}\|\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})-\mathbf{\Phi^0}(\theta_1^a,\theta_2^a,\mathbf{y})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^\infty(\mathscr{F}_0))}\\
&\qquad\le CT\|\theta^a-\theta^b\|_{\Theta_T}.
\end{aligned}
$$

□

Annexe B

The linearized terms

In what follows we give the explicit expression of the functions $\mathbf{L}_1$–$\mathbf{L}_4$ and of the constants $\mathbf{L}_5$–$\mathbf{L}_6$. We denote $\mathbf{t}_s = \mathbf{n}_s^\perp = (-(\mathbf{n}_s)_2, (\mathbf{n}_s)_1)$ a unitary tangent vector to ∂S_s. We have

$$\mathbf{L}_1(\mathbf{y})_i = \Big(\nu \mathbf{L}_{\mathbf{F}^3}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^4}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^5}(\mathbf{y})\Big)_{i1} + (\nabla \mathbf{f}_{\mathscr{F}}(\mathbf{y}) \partial_{\theta_1} \mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))_i, \tag{B.1}$$

$$\mathbf{L}_2(\mathbf{y})_i = \Big(\nu \mathbf{L}_{\mathbf{F}^3}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^4}(\mathbf{y}) + \mathbf{L}_{\mathbf{F}^5}(\mathbf{y})\Big)_{i2} + (\nabla \mathbf{f}_{\mathscr{F}}(\mathbf{y}) \partial_{\theta_2} \mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))_i, \tag{B.2}$$

$$\mathbf{L}_3(\mathbf{y})_i = \Big(\mathbf{L}_{\mathbf{F}^2}(\mathbf{y})\Big)_{i1}, \tag{B.3}$$

$$\mathbf{L}_4(\mathbf{y})_i = \Big(\mathbf{L}_{\mathbf{F}^2}(\mathbf{y})\Big)_{i2}, \tag{B.4}$$

$$\begin{aligned}(\mathbf{L}_5)_i = \int_{\partial S_s} (-\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s) \cdot \partial_{\theta_1\theta_i} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \\ - \sum_{k,\ell} \sigma_F(\mathbf{w}, p_{\mathbf{w}})_{\ell k} (\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{k1} \partial_{\theta_i} \Phi^S_\ell(0,0,\gamma_y) \\ -(\nabla_{\mathbf{y}} \partial_{\theta_1} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\mathbf{t}_s \cdot \mathbf{t}_s) \sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_i} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \\ -\nu \sum_{k,\ell} ((\mathbf{L}_{\mathcal{G}})_{k\ell 1} + (\mathbf{L}_{\mathcal{G}})_{\ell k 1})(\mathbf{n}_s)_k \partial_{\theta_i} \Phi^S_\ell(0,0,\gamma_y) \, \mathrm{d}\gamma_y,\end{aligned} \tag{B.5}$$

$$\begin{aligned}(\mathbf{L}_6)_i = \int_{\partial S_s} (-\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s) \cdot \partial_{\theta_2\theta_i} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \\ - \sum_{k,\ell} \sigma_F(\mathbf{w}, p_{\mathbf{w}})_{\ell k} (\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{k2} \partial_{\theta_i} \Phi^S_\ell(0,0,\gamma_y) \\ -(\nabla_{\mathbf{y}} \partial_{\theta_2} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y)\mathbf{t}_s \cdot \mathbf{t}_s) \sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_i} \mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y) \\ -\nu \sum_{k,\ell} ((\mathbf{L}_{\mathcal{G}})_{k\ell 2} + (\mathbf{L}_{\mathcal{G}})_{\ell k 2})(\mathbf{n}_s)_k \partial_{\theta_i} \Phi^S_\ell(0,0,\gamma_y) \, \mathrm{d}\gamma_y,\end{aligned} \tag{B.6}$$

where the terms used are defined in (B.7)–(B.12). In the previous paragraph, we denoted by $\Big(\mathbf{L}_{\mathbf{F}^k}\Big)_{ij}$ for $3 \leq k \leq 5$, the linearization of $(\mathbf{F}^k)_i$ (defined in (1.66)) around the stationary state $(\mathbf{w}, p_{\mathbf{w}}, 0, 0)$ solution of (2.16) with respect to the parameter θ_j and $\Big(\mathbf{L}_{\mathbf{F}^2}\Big)_{ij}$ is the linearization of $(\mathbf{F}^2)_i$ with respect to $\dot{\theta}_j$. These terms are given below

$$(\mathbf{L}_{\mathbf{F}^2}(\mathbf{y}))_{ij} = \Big((\partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}) \cdot \nabla_{\mathbf{y}})\mathbf{w} + \mathrm{cof}(\nabla_{\mathbf{y}} \partial_{\theta_j} \mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T \mathbf{w}\Big)_i, \tag{B.7}$$

$$
\begin{aligned}
(\mathbf{L}_{\mathbf{F}^3}(\mathbf{y}))_{ij} = & -2\nabla_{\mathbf{y}}\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}) : \nabla^2 w_i(\mathbf{y}) - (\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T\Delta\mathbf{w}(\mathbf{y}))_i \\
& +2\sum_{k,\ell}\mathrm{cof}\left(\frac{\partial}{\partial x_\ell}\left(\nabla_{\mathbf{x}}\partial_{\theta_j}\boldsymbol{\Psi}^{\mathbf{S}}(0,0,\mathbf{y})\right)\right)_{ki}\frac{\partial w_k}{\partial y_\ell}(\mathbf{y}) \\
& +\sum_{\ell,m}\frac{\partial w_i}{\partial y_\ell}(\mathbf{y})\frac{\partial^2}{\partial x_m^2}(\partial_{\theta_j}\boldsymbol{\Psi}^{\mathbf{S}}(0,0,\mathbf{y}))_\ell \\
& +\sum_{k,m}\mathrm{cof}\left(\frac{\partial^2}{\partial x_m^2}\left(\nabla_{\mathbf{x}}\partial_{\theta_j}\boldsymbol{\Psi}^{\mathbf{S}}(0,0,\mathbf{y})\right)\right)_{ki} w_k(\mathbf{y}),
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
(\mathbf{L}_{\mathbf{F}^4}(\mathbf{y}))_{ij} = & -\sum_{k,\ell}\mathrm{cof}\left(\frac{\partial}{\partial x_k}(\nabla_{\mathbf{x}}\partial_{\theta_j}\boldsymbol{\Psi}^{\mathbf{S}}(0,0,\mathbf{y}))\right)_{\ell i} w_k(\mathbf{y})w_\ell(\mathbf{y}) \\
& +\sum_{k,\ell}\Big(2\mathrm{Tr}(\nabla_{\mathbf{y}}\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))\delta_{i\ell} - (\nabla_{\mathbf{y}}\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))_{i\ell}\Big) w_k(\mathbf{y})\frac{\partial w_\ell}{\partial y_k}(\mathbf{y}),
\end{aligned}
\tag{B.9}
$$

and

$$
(\mathbf{L}_{\mathbf{F}^5}(\mathbf{y}))_{ij} = \Big((\nabla_{\mathbf{y}}\partial_{\theta_j}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T\nabla_{\mathbf{y}} p_{\mathbf{w}}(\mathbf{y})\Big)_i. \tag{B.10}
$$

We also define the linearization of $\mathbf{n}_{\theta_1,\theta_2}$ (the unitary outward normal to $\mathscr{F}(\theta_1,\theta_2)$) and $\mathcal{G}$ (defined in (1.67)) with respect to θ_j by

$$
(\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}}(\gamma_y))_{j,n} = (\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\gamma_y))\mathbf{n}_s)_j - (\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\gamma_y))\mathbf{n}_s\cdot\mathbf{n}_s)(\mathbf{n}_s)_j, \tag{B.11}
$$

and

$$
\begin{aligned}
(\mathbf{L}_{\mathcal{G}}(\gamma_y))_{i,j,n} = & \sum_k w_k(\gamma_y)\mathrm{cof}\left(\frac{\partial}{\partial x_j}(\nabla_{\mathbf{x}}\partial_{\theta_n}\boldsymbol{\Psi}^{\mathbf{S}}(0,0,\gamma_y))\right)_{ki} \\
& -\sum_k\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\gamma_y))_{ki}\frac{\partial w_k}{\partial y_j}(\gamma_y) + \sum_k\nabla_{\mathbf{y}}\partial_{\theta_n}\boldsymbol{\Phi}^{\mathbf{S}}(0,0,\gamma_y)_{kj}\frac{\partial w_i}{\partial y_k}(\gamma_y).
\end{aligned}
\tag{B.12}
$$

Annexe C

Independence of the hypothesis $(\mathcal{H})_\delta$ with respect to the diffeomorphism

Let $\mathbf{\Phi}^a(\theta_1,\theta_2,\mathbf{y})$ and $\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y})$ be two diffeomorphisms from $\mathscr{F}_s$ to $\mathscr{F}(\theta_1,\theta_2)$ that are extensions of $\mathbf{X}(\theta_1,\theta_2,\mathbf{y})$ into the fluid domain. We also assume that $\mathbf{\Phi}^a(0,0,\mathbf{y}) = \mathbf{\Phi}^b(0,0,\mathbf{y}) = \mathbf{y}$, $\forall \mathbf{y} \in \mathscr{F}_s$. Let us denote $\mathbf{\Psi}^a(\theta_1,\theta_2,.)$ and $\mathbf{\Psi}^b(\theta_1,\theta_2,.)$ the inverse diffeomorphisms of, respectively, $\mathbf{\Phi}^a(\theta_1,\theta_2,.)$ and $\mathbf{\Phi}^b(\theta_1,\theta_2,.)$. We also denote $\mathcal{J}^a_{\mathbf{\Phi}}(\theta_1,\theta_2,\mathbf{y})$, $\mathcal{J}^b_{\mathbf{\Phi}}(\theta_1,\theta_2,\mathbf{y})$, $\mathcal{J}^a_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{x})$ and $\mathcal{J}^b_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{x})$ the corresponding Jacobian matrices.

We define the difference between the velocity at time t in $\mathscr{F}_s$ and the stationary velocity for each of these diffeomorphisms

$$\forall \mathbf{y} \in \mathscr{F}_s, \quad \forall t \in (0,\infty), \quad \left\{\begin{array}{l} \mathbf{v}^a(t,\mathbf{y}) = \mathrm{cof}(\mathcal{J}^a_{\mathbf{\Phi}}(\theta_1(t),\theta_2(t),\mathbf{y}))^T \mathbf{u}(t,\mathbf{\Phi}^a(\theta_1(t),\theta_2(t),\mathbf{y})) - \mathbf{w}(\mathbf{y}),\\ q^a(t,\mathbf{y}) = p(t,\mathbf{\Phi}^a(\theta_1,\theta_2,\mathbf{y})) - p_{\mathbf{w}}(\mathbf{y}),\\ \mathbf{v}^b(t,\mathbf{y}) = \mathrm{cof}(\mathcal{J}^b_{\mathbf{\Phi}}(\theta_1(t),\theta_2(t),\mathbf{y}))^T \mathbf{u}(t,\mathbf{\Phi}^b(\theta_1(t),\theta_2(t),\mathbf{y})) - \mathbf{w}(\mathbf{y}),\\ q^b(t,\mathbf{y}) = p(t,\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y})) - p_{\mathbf{w}}(\mathbf{y}). \end{array}\right.$$

We have the relations

$$\left\{\begin{array}{l} \mathbf{v}^b(t,\mathbf{y}) = \mathrm{cof}(\mathcal{J}^b_{\mathbf{\Phi}}(\theta_1,\theta_2,\mathbf{y}))^T \mathrm{cof}(\mathcal{J}^a_{\mathbf{\Psi}}(\theta_1,\theta_2,\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y})))^T \Big(\mathbf{v}^a\Big(t,\mathbf{\Psi}^a(\theta_1,\theta_2,\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))\Big)\\ \qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{w}\Big(\mathbf{\Psi}^a(\theta_1,\theta_2,\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))\Big)\Big) - \mathbf{w}(\mathbf{y}),\\ q^b(t,\mathbf{y}) = q^a(t,\mathbf{\Psi}^a(\theta_1,\theta_2,\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))) + p_{\mathbf{w}}(\mathbf{\Psi}^a(\theta_1,\theta_2,\mathbf{\Phi}^b(\theta_1,\theta_2,\mathbf{y}))) - p_{\mathbf{w}}(\mathbf{y}). \end{array}\right.$$

Of course, $(\mathbf{v}^a, q^a, \theta_1, \theta_2)$ satisfies the nonlinear system (2.65)–(2.66) corresponding to the diffeomorphisms $\mathbf{\Phi}^a(\theta_1,\theta_2,.)$ and $\mathbf{\Psi}^a(\theta_1,\theta_2,.)$. Similarly, $(\mathbf{v}^b, q^b, \theta_1, \theta_2)$ fulfils (2.65)–(2.66) given by $\mathbf{\Phi}^b(\theta_1,\theta_2,.)$ and $\mathbf{\Psi}^b(\theta_1,\theta_2,.)$.

We are interested in the following system

$$\left\{\begin{array}{ll} \dfrac{\partial \mathbf{v}^\alpha_L}{\partial t} + (\mathbf{w}\cdot\nabla)\mathbf{v}^\alpha_L + (\mathbf{v}^\alpha_L\cdot\nabla)\mathbf{w} - \mathbf{L}^\alpha_{\mathbf{F}}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2,\mathbf{y}) - \nu\Delta\mathbf{v}^\alpha_L + \nabla q^\alpha_L = 0 & \text{in } (0,\infty)\times\mathscr{F}_s,\\ \mathrm{div}\ \mathbf{v}^\alpha_L = 0 & \text{in } (0,\infty)\times\mathscr{F}_s,\\ \mathbf{v}^\alpha_L = \dot\theta_1\partial_{\theta_1}\mathbf{\Phi}^\alpha(0,0,.) + \dot\theta_2\partial_{\theta_2}\mathbf{\Phi}^\alpha(0,0,.) & \text{on } (0,\infty)\times\partial S_s,\\ \mathbf{v}^\alpha_L = 0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{D}},\\ \sigma_F(\mathbf{v}^\alpha_L, q^\alpha_L)\mathbf{n} = 0 & \text{on } (0,\infty)\times\Gamma_{\mathrm{N}},\\ \mathcal{M}_{\xi_1,\xi_2}\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} = \Big(\displaystyle\int_{\partial S_s}\Big(q^\alpha_L \mathrm{I} - \nu(\nabla\mathbf{v}^\alpha_L + (\nabla\mathbf{v}^\alpha_L)^T)\Big)\mathbf{n}_s\cdot\partial_{\theta_j}\mathbf{\Phi}^\alpha(0,0,\gamma_y)\,\mathrm{d}\gamma_y\Big)_{j=1,2} & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathbf{L}^\alpha_{\mathbf{S}}(\theta_1,\theta_2) & \text{on } (0,\infty), \end{array}\right. \tag{C.1}$$

where $\alpha = a$ or b, and $\mathbf{L}^\alpha_\mathbf{F}$ and $\mathbf{L}^\alpha_\mathbf{S}$ are given respectively by (2.31) and (2.32) corresponding to the choice of diffeomorphism $\mathbf{\Phi}^\mathbf{S}(\theta_1, \theta_2, .) = \mathbf{\Phi}^\alpha(\theta_1, \theta_2, .)$.

We can then show that $(\mathbf{v}^a_L, q^a_L)$ fulfils the linear system (C.1) with $\alpha = a$ if and only if

$$\begin{cases} \mathbf{v}^b_L(t,\mathbf{y}) = \mathbf{v}^a_L(t,\mathbf{y}) + \sum_j \theta_j \Big(\operatorname{cof}(\partial_{\theta_j} \mathcal{J}^b_\mathbf{\Phi}(0,0,\mathbf{y}))^T \mathbf{w}(\mathbf{y}) + \operatorname{cof}(\partial_{\theta_j} \mathcal{J}^a_\mathbf{\Psi}(0,0,\mathbf{y}))^T \mathbf{w}(\mathbf{y}) \\ \qquad\qquad + \nabla \mathbf{w} \times (\partial_{\theta_j} \mathbf{\Psi}^a(0,0,\mathbf{y}) + \partial_{\theta_j} \mathbf{\Phi}^b(0,0,\mathbf{y})) \Big), \\ q^b_L(t,\mathbf{y}) = q^a_L(t,\mathbf{y}) + \sum_j \theta_j \nabla p_\mathbf{w} \cdot (\partial_{\theta_j} \mathbf{\Phi}^a(0,0,\mathbf{y}) - \partial_{\theta_j} \mathbf{\Phi}^b(0,0,\mathbf{y})), \end{cases} \tag{C.2}$$

fulfils the linear system (C.1) with $\alpha = b$.

The proof is a direct computation, for instance

$$\begin{aligned} \frac{\partial \mathbf{v}^b_L}{\partial t} - \mathbf{L}^b_3 \dot\theta_1 - \mathbf{L}^b_4 \dot\theta_2 &= \frac{\partial \mathbf{v}^a_L}{\partial t} + \sum_j \dot\theta_j \Big(\operatorname{cof}\Big(\partial_{\theta_j} \mathcal{J}^b_\mathbf{\Phi}(0,0,\mathbf{y}) - \partial_{\theta_j} \mathcal{J}^a_\mathbf{\Phi}(0,0,\mathbf{y}) \Big)^T \mathbf{w} \\ &\qquad + \nabla \mathbf{w} \times \Big(\partial_{\theta_j} \mathbf{\Phi}^b(0,0,\mathbf{y}) - \partial_{\theta_j} \mathbf{\Phi}^a(0,0,\mathbf{y}) \Big) \Big) - \mathbf{L}^b_3 \dot\theta_1 - \mathbf{L}^b_4 \dot\theta_2 \\ &= \frac{\partial \mathbf{v}^a_L}{\partial t} - \mathbf{L}^a_3 \dot\theta_1 - \mathbf{L}^a_4 \dot\theta_2. \end{aligned}$$

Moreover by using $\operatorname{div} \mathbf{w} = 0$, $\operatorname{cof}(\partial_{\theta_k} \mathcal{J}_\mathbf{\Phi}(0,0,\mathbf{y}))^T : \nabla \mathbf{w} = \partial_{\theta_k} \mathcal{J}_\mathbf{\Psi}(0,0,\mathbf{y}) : \nabla \mathbf{w} = -\partial_{\theta_k} \mathcal{J}_\mathbf{\Phi}(0,0,\mathbf{y}) : \nabla \mathbf{w}$ and the Piola identity, we get

$$\operatorname{div} \mathbf{v}^b_L = \operatorname{div} \mathbf{v}^a_L = 0.$$

This implies that the solution $(\mathbf{v}^b_L, q^b_L, \theta_1, \theta_2)$ to the system (C.1) written with the diffeomorphism $\mathbf{\Phi}^b$ can be derived from the solution to the system (C.1) written with the diffeomorphism $\mathbf{\Phi}^a$ via the relation (C.2). Then the stabilizability of (2.30) is independent with respect to the choice of the diffeomorphism $\mathbf{\Phi}$. As the Hautus test $(\boldsymbol{\mathcal{H}})_\delta$ is equivalent to the stabilizability of (2.30), the hypothesis $(\boldsymbol{\mathcal{H}})_\delta$ is independent from the choice of $\mathbf{\Phi}$.

Annexe D

Proof of Lemma 2.3.2

We first prove some technical lemmas that are used later to decompose the intricate terms of Lemma 2.3.2.

D.1 Technical lemmas

Lemma D.1.1. *Let $R_0 > 0$ be small enough, then there exists a constant K, such that for every $R \leq R_0$ and every $(.,.,\theta_1^j,\theta_2^j)$ in $\widetilde{\mathbb{B}_R^\infty}$, the following estimates hold*

$$\left\| e^{\delta t}\Big(\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a)-\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b)-\sum_n(\theta_n^a-\theta_n^b)\partial_{\theta_n}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.1}$$

$$\left\| e^{\delta t}\Big(\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1^a,\theta_2^a)-\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1^b,\theta_2^b)-\sum_n(\theta_n^a-\theta_n^b)\nabla_{\mathbf{y}}\partial_{\theta_n}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.2}$$

$$\begin{aligned}\Big\| e^{\delta t}\Big(&\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a)-\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b)\\ &+\sum_j(\theta_j^a-\theta_j^b)\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty},\end{aligned} \tag{D.3}$$

$$\begin{aligned}\Big\| e^{\delta t}\Big(&\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a)-\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b)\\ &-\sum_n(\theta_n^a-\theta_n^b)\partial_{x_j}\nabla_{\mathbf{x}}\partial_{\theta_n}\mathbf{\Psi}^{\mathbf{S}}(0,0,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty},\end{aligned} \tag{D.4}$$

$$\begin{aligned}\Big\| e^{\delta t}\Big(&\partial_{x_j}^2\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^a,\theta_2^a)\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a)-\partial_{x_j}^2\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^b,\theta_2^b)\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b)\\ &-\sum_n(\theta_n^a-\theta_n^b)\partial_{x_j}^2\nabla_{\mathbf{x}}\partial_{\theta_n}\mathbf{\Psi}^{\mathbf{S}}(0,0,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty},\end{aligned} \tag{D.5}$$

$$\Big\| e^{\delta t}\Big(\mathbf{n}_{\theta_1^a,\theta_2^a}(\mathbf{\Phi^S}(\theta_1^a,\theta_2^a))_j - \mathbf{n}_{\theta_1^b,\theta_2^b}(\mathbf{\Phi^S}(\theta_1^b,\theta_2^b))_j - \sum_n(\theta_n^a-\theta_n^b)(\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{j,n}\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathrm{L}^\infty(\partial S_s))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.6}$$

$$\Big\| e^{\delta t}\Big(\det(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1^a,\theta_2^a,\mathbf{\Phi^S}{}^a)) - \det(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1^b,\theta_2^b,\mathbf{\Phi^S}{}^b)) + \sum_n(\theta_n^a-\theta_n^b)\mathrm{Tr}(\nabla_{\mathbf{y}}\partial_{\theta_n}\mathbf{\Phi^S}(0,0,.))\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.7}$$

$$\Big\| e^{\delta t}\Big(|\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1^a,\theta_2^a)\mathbf{t}_s| - |\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1^b,\theta_2^b)\mathbf{t}_s| - \sum_k(\theta_k^a-\theta_k^b)\nabla_{\mathbf{y}}\partial_{\theta_k}\mathbf{\Phi^S}(0,0,.)\mathbf{t}_s\cdot\mathbf{t}_s\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\partial S_s))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.8}$$

where $\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}}$ *is defined in* (B.11) *and*

$$\Big\| e^{\delta t}\Big((\partial_t\mathbf{\Psi^S}(\theta_1^a,\theta_2^a))\circ\mathbf{\Phi^S}(\theta_1^a,\theta_2^a) - (\partial_t\mathbf{\Psi^S}(\theta_1^b,\theta_2^b))\circ\mathbf{\Phi^S}(\theta_1^b,\theta_2^b) + \sum_j(\dot\theta_j^a-\dot\theta_j^b)\partial_{\theta_j}\mathbf{\Phi^S}(0,0,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.9}$$

$$\Big\| e^{\delta t}\Big(\partial_t(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1^a,\theta_2^a))\circ\mathbf{\Phi^S}(\theta_1^a,\theta_2^a) - \partial_t(\mathcal{J}_{\mathbf{\Psi^S}}(\theta_1^b,\theta_2^b))\circ\mathbf{\Phi^S}(\theta_1^b,\theta_2^b) + \sum_j(\dot\theta_j^a-\dot\theta_j^b)\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi^S}(0,0,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.10}$$

and

$$\Big\| e^{\delta t}\Big(\mathcal{M}_{\theta_1^a,\theta_2^a} - \mathcal{M}_{\theta_1^b,\theta_2^b}\Big)\Big\|_{\mathrm{L}^\infty(0,\infty)} \leq K\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.11}$$

$$\Big\| e^{\delta t}\Big(\partial_{\theta_j}\mathbf{\Phi^S}(\theta_1^a,\theta_2^a,.) - \partial_{\theta_j}\mathbf{\Phi^S}(\theta_1^b,\theta_2^b,.) - \sum_k(\theta_k^a-\theta_k^b)\partial_{\theta_j\theta_k}\mathbf{\Phi^S}(0,0,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq K\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.12}$$

$$\Big\| e^{\delta t}\Big(\partial_{\theta_j\theta_k}\mathbf{\Phi^S}(\theta_1^a,\theta_2^a,.) - \partial_{\theta_j\theta_k}\mathbf{\Phi^S}(\theta_1^b,\theta_2^b,.)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq K\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.13}$$

$$\Big\| e^{\delta t}\Big(\partial_t\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1^a,\theta_2^a) - \partial_t\mathcal{J}_{\mathbf{\Phi^S}}(\theta_1^b,\theta_2^b)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq K\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}. \tag{D.14}$$

Moreover for every $(\mathbf{v}^j,.,\theta_1^j,\theta_2^j)$ *in* $\widetilde{\mathbb{B}_R^\infty}$, *we have*

$$\Big\| e^{\delta t}\Big(\nabla\mathbf{v}^a - \mathcal{G}(\theta_1^a,\theta_2^a,\mathbf{v}^a) - \nabla\mathbf{v}^b + \mathcal{G}(\theta_1^b,\theta_2^b,\mathbf{v}^b)\Big)\Big\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\partial S_s))} \leq KR(\|\theta^a-\theta^b\|_{\Theta_\delta^\infty} + \|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}_\delta^\infty}), \tag{D.15}$$

$$\Big\| e^{\delta t}\Big(\mathcal{G}(\theta_1^a,\theta_2^a,\mathbf{w})_{ij} - \mathcal{G}(\theta_1^b,\theta_2^b,\mathbf{w})_{ij} - \sum_n(\theta_n^a-\theta_n^b)(\mathbf{L}_{\mathcal{G}})_{i,j,n}\Big)\Big\|_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\partial S_s))} \leq KR\|\theta^a-\theta^b\|_{\Theta_\delta^\infty}, \tag{D.16}$$

where $\mathbf{L}_{\mathcal{G}}$ *is defined in* (B.12) *and* $\mathcal{G}$ *is defined in* (2.70).

A direct application of Lemma D.1.1 obtained by taking $(\theta_1^b, \theta_2^b) = (0,0)$ is the following lemma.

Lemma D.1.2. *Let $R_0 > 0$ be small enough, then there exists $K > 0$ such that for every $R \leq R_0$ and for every $(.,.,\theta_1,\theta_2) \in \widetilde{\mathbb{B}_R^\infty}$, the following estimates hold*

$$\left\| e^{\delta t}\Big(\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1,\theta_2) - \mathrm{I}\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR, \tag{D.17}$$

$$\left\| e^{\delta t}\Big(\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2,\mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2)) - \mathrm{I}\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR, \tag{D.18}$$

$$\left\| e^{\delta t}\Big(\partial_{x_j}\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2) \circ \mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2)\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^1(\Omega))} \leq KR, \tag{D.19}$$

$$\left\| e^{\delta t}\Big(\partial^2_{x_j}\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2) \circ \mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2)\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^2(\Omega))} \leq KR, \tag{D.20}$$

$$\left\| e^{\delta t}\Big(\mathcal{M}_{\theta_1,\theta_2} - \mathcal{M}_{\xi_1,\xi_2}\Big)\right\|_{\mathrm{L}^\infty(0,\infty)} \leq KR, \tag{D.21}$$

$$\left\| e^{\delta t}\Big(\mathbf{n}_{\theta_1,\theta_2}(\mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2)) - \mathbf{n}_s\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\partial S_s))} \leq KR, \tag{D.22}$$

$$\left\| e^{\delta t}\Big(|\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1,\theta_2)\mathbf{t}_s| - 1\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\partial S_s))} \leq KR, \tag{D.23}$$

$$\left\| e^{\delta t}\Big(\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2,.) - \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^3(\Omega))} \leq KR, \tag{D.24}$$

and

$$\|e^{\delta t}\partial_t\mathcal{J}_{\mathbf{\Phi}^{\mathbf{S}}}(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,\infty;\mathbf{H}^2(\Omega))} \leq KR, \tag{D.25}$$

$$\|e^{\delta t}(\partial_t\mathbf{\Psi}^{\mathbf{S}}(\theta_1,\theta_2)) \circ \mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR, \tag{D.26}$$

$$\|e^{\delta t}(\partial_t\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1,\theta_2)) \circ \mathbf{\Phi}^{\mathbf{S}}(\theta_1,\theta_2)\|_{\mathrm{L}^\infty(0,\infty;\mathbf{L}^\infty(\Omega))} \leq KR, \tag{D.27}$$

and

$$\left\| e^{\delta t}\Big(\nabla\mathbf{w} - \mathcal{G}(\theta_1,\theta_2,\mathbf{w})\Big)\right\|_{\mathrm{L}^2(0,\infty;\mathbf{L}^2(\partial S_s))} \leq KR, \tag{D.28}$$

where $\mathcal{G}$ is defined in (2.70).

Proof of Lemma D.1.1. Let $\mathbb{X}$ be some Banach space. All functions in Lemma D.1.1 can be written as $\alpha = \alpha(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)$ valued in $\mathbb{X}$. The proof of estimates (D.11)–(D.14) uses the mean value theorem,

$$\begin{aligned}
&\left\| e^{\delta t}\Big(\alpha(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a) - \alpha(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\Big)\right\|_{\mathrm{L}^\infty(0,\infty;\mathbb{X})} \\
&\quad \leq \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta \\ |\omega_1|+|\omega_2|\leq R}} \|\nabla_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{\mathrm{L}^\infty(0,\infty;\mathbb{X})}\Big(\|e^{\delta t}(\theta^a - \theta^b)\|_{\mathrm{L}^\infty(0,\infty)} + \|e^{\delta t}(\dot\theta^a - \dot\theta^b)\|_{\mathrm{L}^\infty(0,\infty)}\Big) \\
&\quad \leq 2C \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta \\ |\omega_1|+|\omega_2|\leq R}} \|\nabla_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{\mathrm{L}^\infty(0,\infty;\mathbb{X})}\|\theta^a - \theta^b\|_{\Theta_\delta^\infty}.
\end{aligned}$$

To prove the other estimates (D.1)–(D.10) and (D.16), we do a Taylor expansion of order 2 and use the mean value theorem. We get

$$\Big\| e^{\delta t}\Big(\alpha(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\alpha(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)-\sum_{j=1}^{2}\partial_{\theta_j}\alpha(0,0,0,0)(\theta_j^a-\theta_j^b)-\sum_{j=1}^{2}\partial_{\omega_j}\alpha(0,0,0,0)(\dot\theta_j^a-\dot\theta_j^b)\Big)\Big\|_{\mathrm{L}^\infty(0,\infty;\mathbb{X})}$$
$$\le \sup_{\substack{(\theta_1,\theta_2)\in\mathbb{D}_\Theta\\ |\omega_1|+|\omega_2|\le R}} \|\nabla^2_{\theta,\omega}\alpha(\theta_1,\theta_2,\omega_1,\omega_2)\|_{\mathrm{L}^\infty(0,\infty;\mathbb{X})} R(\|e^{\delta t}(\theta^a-\theta^b)\|_{\mathrm{L}^\infty(0,\infty)}+\|e^{\delta t}(\dot\theta^a-\dot\theta^b)\|_{\mathrm{L}^\infty(0,\infty)}).$$

The estimate (D.15) is proven by using the previous estimates. □

D.2 Detailed proof of Lemma 2.3.2

Proof. The weights in time $e^{\delta t}$ of the nonlinear terms can be easily handled. All the difficulties then come from space regularity issues, which can be handled as in Appendix A for the terms $\mathbf{F}(\theta_1^a,\theta_2^a,\mathbf{v}^a,q^a)-\mathbf{F}(\theta_1^b,\theta_2^b,\mathbf{v}^b,q^b)$, $\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)$ and $\mathbf{S}(\theta_1^a,\theta_2^a,\mathbf{v}^a,q^a)-\mathbf{S}(\theta_1^b,\theta_2^b,\mathbf{v}^b,q^b)$ using the estimates of Lammas D.1.1 and D.1.2 :

$$\|\mathbf{F}(\theta_1^a,\theta_2^a,\mathbf{v}^a,q^a)-\mathbf{F}(\theta_1^b,\theta_2^b,\mathbf{v}^b,q^b)\|_{\mathbb{F}^\infty_\delta}\le KR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}^\infty_\delta}+\|q^a-q^b\|_{\mathbb{P}^\infty_\delta}+\|\theta^a-\theta^b\|_{\Theta^\infty_\delta}),$$

$$\|\mathbf{G}(\theta_1^a,\theta_2^a,\dot\theta_1^a,\dot\theta_2^a)-\mathbf{G}(\theta_1^b,\theta_2^b,\dot\theta_1^b,\dot\theta_2^b)\|_{\mathbb{G}^\infty_\delta}\le KR\|\theta^a-\theta^b\|_{\Theta^\infty_\delta},$$

and

$$\|\mathbf{S}(\theta_1^a,\theta_2^a,\mathbf{v}^a,q^a)-\mathbf{S}(\theta_1^b,\theta_2^b,\mathbf{v}^b,q^b)\|_{\mathbb{S}^\infty_\delta}\le KR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}^\infty_\delta}+\|q^a-q^b\|_{\mathbb{P}^\infty_\delta}+\|\theta^a-\theta^b\|_{\Theta^\infty_\delta}),$$

where the terms $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{S}$ are defined in (2.68)–(2.69).

Now, it remains to prove

$$\begin{aligned}\|\mathbf{F}(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a,p_\mathbf{w}+q^a)&-\mathbf{F}(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b,p_\mathbf{w}+q^b)-\mathbf{F}(\theta_1^a,\theta_2^a,\mathbf{v}^a,q^a)+\mathbf{F}(\theta_1^b,\theta_2^b,\mathbf{v}^b,q^b)\\ &+(\mathbf{w}\cdot\nabla)(\mathbf{v}^a-\mathbf{v}^b)+((\mathbf{v}^a-\mathbf{v}^b)\cdot\nabla)\mathbf{w}\\ &-\mathbf{L}_\mathbf{F}(\theta_1^a-\theta_1^b,\theta_2^a-\theta_2^b,\dot\theta_1^a-\dot\theta_1^b,\dot\theta_2^a-\dot\theta_2^b)+\mathbf{f}_\mathscr{F}(\mathbf{\Phi}^\mathbf{S}(\theta_1^a,\theta_2^a))-\mathbf{f}_\mathscr{F}(\mathbf{\Phi}^\mathbf{S}(\theta_1^b,\theta_2^b))\|_{\mathbb{F}^\infty_\delta}\\ &\le CR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}^\infty_\delta}+\|\theta^a-\theta^b\|_{\Theta^\infty_\delta}),\end{aligned} \tag{D.29}$$

and

$$\begin{aligned}\|\mathbf{S}(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a,p_\mathbf{w}+q^a)-\mathbf{S}(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b,p_\mathbf{w}+q^b)-\mathbf{S}(\theta_1^a,\theta_2^a,\mathbf{v}^a,q^a)+\mathbf{S}(\theta_1^b,\theta_2^b,\mathbf{v}^b,q^b)\\ -\mathbf{L}_\mathbf{S}(\theta_1^a-\theta_1^b,\theta_2^a-\theta_2^b)\|_{\mathbb{S}^\infty_\delta}\le CR\|\theta^a-\theta^b\|_{\Theta^\infty_\delta},\end{aligned} \tag{D.30}$$

and then the proof will be complete.

The estimate (D.29) is a consequence of the following relations

$$\mathbf{F}^1(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^1(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^1(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^1(\theta_1^b,\theta_2^b,\mathbf{v}^b)=0, \tag{D.31}$$

$$\begin{aligned}\Big\|&\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{v}^b)\\ &-\sum_j(\dot\theta_j^a-\dot\theta_j^b)\Big((\partial_{\theta_j}\mathbf{\Phi}^\mathbf{S}(0,0,\mathbf{y})\cdot\nabla_\mathbf{y})\mathbf{w}+\mathrm{cof}(\nabla_\mathbf{y}\partial_{\theta_j}\mathbf{\Phi}^\mathbf{S}(0,0,\mathbf{y}))^T\mathbf{w}\Big)\Big\|_{\mathbb{F}^\infty_\delta}\le KR\|\theta^a-\theta^b\|_{\Theta^\infty_\delta},\end{aligned} \tag{D.32}$$

$$
\begin{aligned}
&\Big\|\big(\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\big)_i\\
&\qquad\qquad -\nu\sum_j(\theta_j^a-\theta_j^b)(\mathbf{L}_{\mathbf{F}^3})_{ij}\Big\|_{\mathbb{F}^\infty_\delta}\le KR\|\theta^a-\theta^b\|_{\Theta^\infty_\delta},
\end{aligned}
\tag{D.33}
$$

$$
\begin{aligned}
&\Big\|\big(\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^4(\theta_1^a,\theta_2^a,\mathbf{v}^a)-\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)+\mathbf{F}^4(\theta_1^b,\theta_2^b,\mathbf{v}^b)\\
&\qquad +((\mathbf{v}^a-\mathbf{v}^b)\cdot\nabla)\mathbf{w}+(\mathbf{w}\cdot\nabla)(\mathbf{v}^a-\mathbf{v}^b)\big)_i-\sum_n(\theta_n^a-\theta_n^b)(\mathbf{L}_{\mathbf{F}^4})_{i,n}\Big\|_{\mathbb{F}^\infty_\delta}\\
&\qquad\le KR(\|\mathbf{v}^a-\mathbf{v}^b\|_{\mathbb{U}^\infty_\delta}+\|\theta^a-\theta^b\|_{\Theta^\infty_\delta}),
\end{aligned}
\tag{D.34}
$$

$$
\begin{aligned}
&\Big\|\mathbf{F}^5(\theta_1^a,\theta_2^a,p_{\mathbf{w}}+q^a)-\mathbf{F}^5(\theta_1^a,\theta_2^a,q^a)-\mathbf{F}^5(\theta_1^b,\theta_2^b,p_{\mathbf{w}}+q^b)+\mathbf{F}^5(\theta_1^b,\theta_2^b,q^b)\\
&\qquad -\sum_j(\theta_j^a-\theta_j^b)(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T\nabla_{\mathbf{y}}p_{\mathbf{w}}\Big\|_{\mathbb{F}^\infty_\delta}\le KR\|\theta^a-\theta^b\|_{\Theta^\infty_\delta},
\end{aligned}
\tag{D.35}
$$

and

$$
\begin{aligned}
&\Big\|\mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a))-\mathbf{f}_{\mathscr{F}}(\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b))-\sum_k(\theta_k^a-\theta_k^b)\nabla_{\mathbf{y}}\mathbf{f}_{\mathscr{F}}\partial_{\theta_k}\mathbf{\Phi}^{\mathbf{S}}(0,0,.)\Big\|_{\mathbb{F}^\infty_\delta}\\
&\qquad\le K\|\mathbf{f}_{\mathscr{F}}\|_{\mathbf{W}^{1,\infty}(\Omega)}R\|\theta^a-\theta^b\|_{\Theta^\infty_\delta}.
\end{aligned}
\tag{D.36}
$$

In the sequel, we use the compact notations $\mathcal{J}^a_{\mathbf{\Psi}^{\mathbf{S}}}=\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^a,\theta_2^a,\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a,\mathbf{y}))$ and $\mathcal{J}^b_{\mathbf{\Psi}^{\mathbf{S}}}=\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^b,\theta_2^b,\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b,\mathbf{y}))$, and similarly for other functions. We now prove the estimates (D.30)–(D.36). We keep (D.30) for the end.

• **Identity** (D.31) **:** the proof is immediate.

• **Estimate** (D.32) **:** we use the decomposition

$$
\begin{aligned}
&\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^2(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^2(\theta_1^b,\theta_2^b,\mathbf{v}^b)\\
&\qquad -\sum_j(\dot\theta_j^a-\dot\theta_j^b)\Big((\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y})\cdot\nabla_{\mathbf{y}})\mathbf{w}+\mathrm{cof}(\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T\mathbf{w}\Big)\\
&=-\mathrm{cof}\Big(\partial_t(\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^a,\theta_2^a))\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a,\mathbf{y})-\partial_t(\mathcal{J}_{\mathbf{\Psi}^{\mathbf{S}}}(\theta_1^b,\theta_2^b))\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b,\mathbf{y})\\
&\qquad +\sum_j(\dot\theta_j^a-\dot\theta_j^b)\nabla_{\mathbf{y}}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y})\Big)^T\mathbf{w}\\
&\quad -\mathrm{cof}\Big(\mathcal{J}^a_{\mathbf{\Psi}^{\mathbf{S}}}-\mathcal{J}^b_{\mathbf{\Psi}^{\mathbf{S}}}\Big)^T\nabla_{\mathbf{y}}\mathbf{w}(\partial_t\mathbf{\Psi}^{\mathbf{S}}(\theta_1^a,\theta_2^a,.))\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a,\mathbf{y})\\
&\quad -\mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}^{\mathbf{S}}})^T\nabla_{\mathbf{y}}\mathbf{w}\Big(\partial_t(\mathbf{\Psi}^{\mathbf{S}}(\theta_1^a,\theta_2^a,.))\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^a,\theta_2^a,\mathbf{y})-\partial_t(\mathbf{\Psi}^{\mathbf{S}}(\theta_1^b,\theta_2^b,.))\circ\mathbf{\Phi}^{\mathbf{S}}(\theta_1^b,\theta_2^b,\mathbf{y})\\
&\qquad +\sum_j(\dot\theta_j^a-\dot\theta_j^b)\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y})\Big)\\
&\quad +\sum_j(\dot\theta_j^a-\dot\theta_j^b)\mathrm{cof}\Big(\mathcal{J}^b_{\mathbf{\Psi}^{\mathbf{S}}}-\mathrm{I}\Big)^T\nabla_{\mathbf{y}}\mathbf{w}\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}),
\end{aligned}
$$

and the estimates (D.18), (D.26), (D.3), (D.9) and (D.10) yield (D.32).

• **Estimate** (D.33) **:** we use the decomposition

$$
\begin{aligned}
&\Big(\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{w}+\mathbf{v}^a)-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{w}+\mathbf{v}^b)-\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{v}^a)+\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{v}^b)\Big)_i-\nu\sum_j(\theta_j^a-\theta_j^b)(\mathbf{L}_{\mathbf{F}^3})_{ij}\\
&\qquad =\Big(\mathbf{F}^3(\theta_1^a,\theta_2^a,\mathbf{w})-\mathbf{F}^3(\theta_1^b,\theta_2^b,\mathbf{w})\Big)_i-\nu\sum_j(\theta_j^a-\theta_j^b)(\mathbf{L}_{\mathbf{F}^3})_{ij}\\
&\qquad =A_{1,i}+A_{2,i}+A_{3,i}+A_{4,i},
\end{aligned}
$$

where

$$A_{1,i} = \nu \sum_{j,k,\ell,m} \Bigg(\mathrm{cof}(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial \Psi^a_\ell}{\partial x_j} \frac{\partial \Psi^a_m}{\partial x_j} - \mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial \Psi^b_\ell}{\partial x_j} \frac{\partial \Psi^b_m}{\partial x_j}$$
$$+ \sum_n (\theta^a_n - \theta^b_n) \Big(\partial_{y_j} \partial_{\theta_n} \Phi_\ell(0,0,\mathbf{y}) \delta_{ki} \delta_{mj} + \partial_{y_j} \partial_{\theta_n} \Phi_m(0,0,\mathbf{y}) \delta_{ki} \delta_{\ell j} + \mathrm{cof}(\partial_{\theta_n} \mathcal{J}_{\mathbf{\Phi}\mathbf{s}}(0,0,\mathbf{y}))_{ki} \delta_{mj} \delta_{\ell j} \Big)$$
$$\Bigg) \frac{\partial^2 w_k}{\partial y_m \partial y_\ell},$$

$$A_{2,i} = 2\nu \sum_{j,k,\ell} \Big(\mathrm{cof}(\partial_{x_j} \mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial \Psi^a_\ell}{\partial x_j} - \mathrm{cof}(\partial_{x_j} \mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial \Psi^b_\ell}{\partial x_j} - \sum_n (\theta^a_n - \theta^b_n) \mathrm{cof}(\partial_{x_j} \partial_{\theta_n} \mathcal{J}_{\mathbf{\Psi}\mathbf{s}}(0,0,\mathbf{y}))_{ki} \delta_{\ell j} \Big) \frac{\partial w_k}{\partial y_\ell},$$

$$A_{3,i} = \nu \sum_{j,k,\ell} \Big(\mathrm{cof}(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial^2 \Psi^a_\ell}{\partial x_j^2} - \mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial^2 \Psi^b_\ell}{\partial x_j^2} - \sum_n (\theta^a_n - \theta^b_n) \delta_{ki} \frac{\partial^2}{\partial x_j^2} \partial_{\theta_n} \Psi_\ell(0,0,\mathbf{y}) \Big) \frac{\partial w_k}{\partial y_\ell},$$

and

$$A_{4,i} = \nu \sum_{j,k} \left(\mathrm{cof}\left(\frac{\partial^2}{\partial x_j^2} \mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}} \right)_{ki} - \mathrm{cof}\left(\frac{\partial^2}{\partial x_j^2} \mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}} \right)_{ki} - \sum_n (\theta^a_n - \theta^b_n) \frac{\partial^2}{\partial x_j^2} \partial_{\theta_n} \mathcal{J}_{\mathbf{\Psi}\mathbf{s}}(0,0,\mathbf{y})_{ki} \right) w_k.$$

The term $A_{1,i}$ is of the form $\sum_{j,k,\ell,m} a_{ijk\ell m}(\mathbf{y}) \times \frac{\partial^2 w_k}{\partial y_m \partial y_\ell}$, where

$$a_{ijk\ell m}(\mathbf{y}) = \Bigg(\mathrm{cof}(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial \Psi^a_\ell}{\partial x_j} \frac{\partial \Psi^a_m}{\partial x_j} - \mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{ki} \frac{\partial \Psi^b_\ell}{\partial x_j} \frac{\partial \Psi^b_m}{\partial x_j}$$
$$+ \sum_n (\theta^a_n - \theta^b_n) \Big(\partial_{y_j} \partial_{\theta_n} \Phi_\ell(0,0,\mathbf{y}) \delta_{ki} \delta_{mj} + \partial_{y_j} \partial_{\theta_n} \Phi_m(0,0,\mathbf{y}) \delta_{ki} \delta_{\ell j} + \mathrm{cof}(\partial_{\theta_n} \mathcal{J}_{\mathbf{\Phi}\mathbf{s}}(0,0,\mathbf{y}))_{ki} \delta_{mj} \delta_{\ell j} \Big) \Bigg).$$

Moreover, according to (1.33), $\mathbf{\Psi}^{\mathbf{S}}(\theta_1, \theta_2, \mathbf{y}) = \mathbf{y}$ in $\Omega \backslash \Omega_\varepsilon$, then $a_{ijk\ell m} = 0$ in $\Omega \backslash \Omega_\varepsilon$ (Ω_ε is defined in Lemma 1.1.4). We then estimate $A_{1,i}$ as

$$\left\| a_{ijk\ell m} \times \frac{\partial^2 w_k}{\partial y_m \partial y_\ell} \right\|_{\mathbf{L}^2(\mathscr{F}_s)} \leq \left\| \frac{a_{ijk\ell m}}{\prod_{j \in \mathscr{J}_{d,n}} r_j^\beta} \right\|_{\mathbf{L}^\infty(\mathscr{F}_s)} \left\| \frac{\partial^2 w_k}{\partial y_m \partial y_\ell} \prod_{j \in \mathscr{J}_{d,n}} r_j^\beta \right\|_{\mathbf{L}^2(\mathscr{F}_s)} \leq \| a_{ijk\ell m} \|_{\mathbf{L}^\infty(\Omega)} \left\| \frac{1}{\prod_{j \in \mathscr{J}_{d,n}} r_j^\beta} \right\|_{\mathbf{L}^\infty(\Omega \backslash \Omega_\varepsilon)} \| \mathbf{w} \|_{\mathbf{H}^2_\beta(\mathscr{F}_s)}.$$

Moreover, we get the estimate $\|a_{ijk\ell m}\|_{\mathbf{L}^\infty(\Omega)} \leq KR \|\theta^a - \theta^b\|_{\Theta^\infty_\delta}$ by using three times the estimate (D.3). We then use estimates (D.3), (D.4) and (D.19) for $A_{2,i}$ and $A_{3,i}$ and estimate (D.5) for $A_{4,i}$. We obtain (D.33).

• **Estimate** (D.34) **:** we have the decomposition

$$\Big(\mathbf{F}^4(\theta^a_1, \theta^a_2, \mathbf{w} + \mathbf{v}^a) - \mathbf{F}^4(\theta^a_1, \theta^a_2, \mathbf{v}^a) - \mathbf{F}^4(\theta^b_1, \theta^b_2, \mathbf{w} + \mathbf{v}^b) + \mathbf{F}^4(\theta^b_1, \theta^b_2, \mathbf{v}^b) + ((\mathbf{v}^a - \mathbf{v}^b) \cdot \nabla) \mathbf{w}$$
$$+ (\mathbf{w} \cdot \nabla)(\mathbf{v}^a - \mathbf{v}^b) \Big)_i - \sum_n (\theta^a_n - \theta^b_n) (\mathbf{L}_{\mathbf{F}^4})_{i,n}$$
$$= B_{1,i} + B_{2,i} + B_{3,i} + B_{4,i} + B_{5,i} + B_{6,i},$$

where

$$B_{1,i}=-\sum_{j,k,\ell}\Big(\mathrm{cof}(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{kj}\mathrm{cof}(\partial_{x_j}\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{\ell i}-\mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{kj}\mathrm{cof}(\partial_{x_j}\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{\ell i}$$
$$-\sum_n(\theta^a_n-\theta^b_n)\mathrm{cof}(\partial_{x_j}\partial_{\theta_n}\mathcal{J}_{\mathbf{\Psi}\mathbf{s}}(0,0,\mathbf{y}))_{\ell i}\delta_{kj}\Big)w_kw_\ell,$$
$$B_{2,i}=-\sum_{j,k,\ell}\Big(\mathrm{cof}(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{kj}\mathrm{cof}(\partial_{x_j}\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})_{\ell i}-\mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{kj}\mathrm{cof}(\partial_{x_j}\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{\ell i}\Big)(w_kv^a_\ell+v^a_kw_\ell),$$
$$B_{3,i}=-\sum_{j,k,\ell}\mathrm{cof}(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{kj}\mathrm{cof}(\partial_{x_j}\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})_{\ell i}\Big(w_k(v^a_\ell-v^b_\ell)+(v^a_k-v^b_k)w_\ell\Big),$$
$$B_{4,i}=-\sum_{k,\ell}\Big(\det(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})^2\frac{\partial\Phi^a_i}{\partial y_\ell}-\det(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})^2\frac{\partial\Phi^b_i}{\partial y_\ell}$$
$$+\sum_n(\theta^a_n-\theta^b_n)\Big(2\mathrm{Tr}(\partial_{\theta_n}\mathcal{J}_{\mathbf{\Phi}\mathbf{s}}(0,0,\mathbf{y}))\delta_{i\ell}-\partial_{\theta_n}\mathcal{J}_{\mathbf{\Phi}\mathbf{s}}(0,0,\mathbf{y})_{i\ell}\Big)\Big)w_k\frac{\partial w_\ell}{\partial y_k},$$
$$B_{5,i}=-\sum_{k,\ell}\Big(\det(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}})^2\frac{\partial\Phi^a_i}{\partial y_\ell}-\det(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})^2\frac{\partial\Phi^b_i}{\partial y_\ell}\Big)\Big(\frac{\partial w_\ell}{\partial y_\ell}v^a_k+\frac{\partial v^a_\ell}{\partial y_\ell}w_k\Big),$$

and

$$B_{6,i}=-\sum_{k,\ell}\Big(\det(\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}})^2\frac{\partial\Phi^b_i}{\partial y_\ell}-\delta_{i\ell}\Big)\Big(\frac{\partial w_\ell}{\partial y_k}(v^a_k-v^b_k)+(\frac{\partial v^a_\ell}{\partial y_k}-\frac{\partial v^b_\ell}{\partial y_k})w_k\Big).$$

Now, we use

— estimates (D.3), (D.4), (D.18) and (D.19) for $B_{1,i}$,
— estimates (D.3), (D.4) for $B_{2,i}$,
— estimates (D.18) and (D.19) for $B_{3,i}$,
— estimates (D.2), (D.7), (D.17) and (D.18) for $B_{4,i}$ and $B_{5,i}$,
— estimates (D.17) and (D.18) for $B_{6,i}$.

We get (D.34).

• **Estimate** (D.35) **:** we have

$$\mathbf{F}^5(\theta^a_1,\theta^a_2,p_{\mathbf{w}}+q^a)-\mathbf{F}^5(\theta^a_1,\theta^a_2,q^a)-\mathbf{F}^5(\theta^b_1,\theta^b_2,p_{\mathbf{w}}+q^b)+\mathbf{F}^5(\theta^b_1,\theta^b_2,q^b)$$
$$-\sum_n(\theta^a_n-\theta^b_n)(\nabla_{\mathbf{y}}\partial_{\theta_n}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T\nabla_{\mathbf{y}}p_{\mathbf{w}}$$
$$=\mathbf{F}^5(\theta^a_1,\theta^a_2,p_{\mathbf{w}})-\mathbf{F}^5(\theta^b_1,\theta^b_2,p_{\mathbf{w}})-\sum_n(\theta^a_n-\theta^b_n)(\nabla_{\mathbf{y}}\partial_{\theta_n}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y}))^T\nabla_{\mathbf{y}}p_{\mathbf{w}}$$
$$=-\Big(\mathcal{J}^a_{\mathbf{\Psi}\mathbf{s}}-\mathcal{J}^b_{\mathbf{\Psi}\mathbf{s}}+\sum_n(\theta^a_n-\theta^b_n)\nabla_{\mathbf{y}}\partial_{\theta_n}\mathbf{\Phi}^{\mathbf{S}}(0,0,\mathbf{y})\Big)^T\nabla_{\mathbf{y}}p_{\mathbf{w}},$$

and estimate (D.3) gives (D.35).

• **Estimate** (D.36) **:** it is a consequence of (D.1).

• **Estimate** (D.30) **:** we have

$$\Big(\mathbf{S}(\theta^a_1,\theta^a_2,\mathbf{w}+\mathbf{v}^a,p_{\mathbf{w}}+q^a)-\mathbf{S}(\theta^a_1,\theta^a_2,\mathbf{v}^a,q^a)-\mathbf{S}(\theta^b_1,\theta^b_2,\mathbf{w}+\mathbf{v}^b,p_{\mathbf{w}}+q^b)$$
$$+\mathbf{S}(\theta^b_1,\theta^b_2,\mathbf{v}^b,q^b)-\mathbf{L}_{\mathbf{S}}(\theta^a_1-\theta^b_1,\theta^a_2-\theta^b_2)\Big)_j$$
$$=D_{1,j}+D_{2,j}+D_{3,j}+D_{4,j},$$

where

$$
\begin{aligned}
D_{1,j} = &\int_{\partial S_s} (|\mathcal{J}^a_{\mathbf{\Phi^S}}\mathbf{t}_s| - |\mathcal{J}^b_{\mathbf{\Phi^S}}\mathbf{t}_s|)\Big(p_{\mathbf{w}}\mathrm{I} - \nu(\mathcal{G}^a + (\mathcal{G}^a)^T)\Big)(\mathbf{n}^a_{\theta_1,\theta_2} \circ \mathbf{\Phi}^{\mathbf{S}^a}) \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}^a} \\
&\qquad + \sum_n (\theta^a_n - \theta^b_n) \int_{\partial S_s} ((\nabla_{\mathbf{y}}\partial_{\theta_n}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y))\mathbf{t}_s \cdot \mathbf{t}_s)\sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y), \\
D_{2,j} = &-\nu \int_{\partial S_s} |\mathcal{J}^b_{\mathbf{\Phi^S}}\mathbf{t}_s|\Big(\mathcal{G}^a + (\mathcal{G}^a)^T - \mathcal{G}^b - (\mathcal{G}^b)^T\Big)(\mathbf{n}^a_{\theta_1,\theta_2} \circ \mathbf{\Phi}^{\mathbf{S}^a}) \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}^a} \\
&\qquad + \nu \sum_{k,\ell,n} (\theta^a_n - \theta^b_n) \int_{\partial S_s} ((\mathbf{L}_{\mathcal{G}})_{k\ell n} + (\mathbf{L}_{\mathcal{G}})_{\ell k n})(\mathbf{n}_s)_k \partial_{\theta_j}\Phi_\ell(0,0,\gamma_y), \\
D_{3,j} = &\int_{\partial S_s} |\mathcal{J}^b_{\mathbf{\Phi^S}}\mathbf{t}_s|\Big(p_{\mathbf{w}}\mathrm{I} - \nu(\mathcal{G}^b + (\mathcal{G}^b)^T)\Big)(\mathbf{n}^a_{\theta_1,\theta_2} \circ \mathbf{\Phi}^{\mathbf{S}^a} - \mathbf{n}^b_{\theta_1,\theta_2} \circ \mathbf{\Phi}^{\mathbf{S}^b}) \cdot \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}^a} \\
&\qquad + \sum_{k,\ell,n} (\theta^a_n - \theta^b_n) \int_{\partial S_s} \sigma_F(\mathbf{w}, p_{\mathbf{w}})_{\ell k} (\mathbf{L}_{\mathbf{n}_{\theta_1,\theta_2}})_{kn} \partial_{\theta_j}\Phi_\ell(0,0,\gamma_y),
\end{aligned}
$$

and

$$
\begin{aligned}
D_{4,j} = &\int_{\partial S_s} |\mathcal{J}^b_{\mathbf{\Phi^S}}\mathbf{t}_s|\Big(p_{\mathbf{w}}\mathrm{I} - \nu(\mathcal{G}^b + (\mathcal{G}^b)^T)\Big)(\mathbf{n}^b_{\theta_1,\theta_2} \circ \mathbf{\Phi}^{\mathbf{S}^b}) \cdot (\partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}^a} - \partial_{\theta_j}\mathbf{\Phi}^{\mathbf{S}^b}) \\
&\qquad + \sum_n (\theta^a_n - \theta^b_n) \int_{\partial S_s} \sigma_F(\mathbf{w}, p_{\mathbf{w}})\mathbf{n}_s \cdot \partial_{\theta_n\theta_j}\mathbf{\Phi}^{\mathbf{S}}(0,0,\gamma_y).
\end{aligned}
$$

We use
— estimates (D.8), (D.28), (D.22) and (D.24) for $D_{1,j}$,
— estimates (D.23), (D.16), (D.22) and (D.24) for $D_{2,j}$,
— estimates (D.23), (D.28), (D.6) and (D.24) for $D_{3,j}$,
— estimates (D.23), (D.28), (D.22) and (D.12) for $D_{4,j}$.
Combining these estimates yields (D.30). □

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Étude d'un problème d'interaction fluide–structure : Modélisation, Analyse, Stabilisation et Simulations numériques

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Résumé : Ce travail de thèse porte sur l'étude d'un système d'interaction fluide–structure. Nous en traitons de nombreux aspects allant de sa modélisation jusqu'à l'étude de sa stabilisation et de sa simulation numérique.

Le premier chapitre du manuscrit aborde la modélisation du système ainsi que l'existence de solutions fortes en temps petits. Le fluide est représenté par les équations de Navier–Stokes incompressibles. La structure est déformable et dépend d'un nombre fini de paramètres. Nous obtenons ses équations en appliquant un principe des travaux virtuels. Le système d'équations final est non linéaire.

Nous prouvons l'existence locale d'une solution à ce système, dans un premier temps sur le système linéarisé autour de l'état nul. Puis, nous prouvons l'existence de solutions en temps petits au système non linéaire grâce à un argument de point fixe.

Le deuxième chapitre traite de la stabilisation par feedback autour d'un état stationnaire non nul du système présenté dans le Chapitre 1. L'opérateur de feedback est déterminé à partir de l'analyse du problème linéarisé autour de l'état stationnaire et de la résolution d'une équation de Riccati. Le résultat de stabilisation portant sur le système non linéaire requiert des données petites et est obtenu par un argument de point fixe.

Le troisième chapitre se concentre sur les aspects numériques de ce problème. La construction de l'opérateur de feedback correspond à la version discrétisée de celle proposée dans le Chapitre 2. Le système fluide–structure est simulé en utilisant une méthode de domaines fictifs.
Mots–clés : équations de Navier–Stokes, interaction fluide–structure, stabilisation, méthodes numériques, domaines fictifs.

Abstract : This PhD thesis deals with the study of a fluid–structure interaction system. We are interested in several aspects such as modelling, stabilization and numerical simulation.

In the first chapter of the manuscript, we show the modelling of the system and prove the existence of strong solutions in small times. The fluid is modelled by the incompressible Navier–Stokes equations. The structure is deformable and depends on a finite number of parameters. The equations are obtained with a virtual work principle. The final system of equations is nonlinear.

We prove local existence of a solution to this system, first on the linearized system. Then, existence of solutions in small times to the full nonlinear system is obtained with a fixed point argument.

In the second chapter, we prove feedback stabilization of the problem around a non–null stationary state. The feedback operator is computed with the solution to a Riccati equation obtained by the analysis of the linearized problem around the stationary state. The stabilization result holds on the full nonlinear system and requires small data. It is proven by a fixed point argument.

In the third chapter, we focus on the numerical aspects of the problem. The feedback operator used corresponds to a discretization of the feedback operator of Chapter 2. The solution to the full nonlinear system is computed by the use of a fictitious domain method.
Keywords : Navier–Stokes equations, fluid–structure interaction, stabilization, numerical Methods, fictitious domains.